Community Formation in Networks*

Olivier Bochet† Romain Ferrali‡ Yves Zenou§

July 12, 2024

Abstract

We consider a community formation process on a network. Following a seed agent, the process unfolds sequentially, constrained by the network, and achieved through a sequence of recruited agents making strategic offers to their neighbors. Under arbitrary, homogeneous payoffs, there is essentially a unique subgame perfect equilibrium that is seed-optimal (i.e., maximizing the seed's payoff). Considering heterogeneous payoffs, we highlight a tension with efficiency and identify conditions that restore seed-optimality. Specializing to payoffs monotonic in community and neighborhood sizes, we characterize equilibrium communities across several important economic applications. We finally investigate key players and how denser networks influence equilibrium welfare.

Keywords: Communities, networks, seeds, key players, network density, welfare.
JEL Classification: C72, D85, Z13.

*We thank Renaud Bourlès, Yann Bramoullé, and Ludovic Renou for their helpful comments. Olivier Bochet acknowledges financial support from Tamkeen under the NYU Abu Dhabi Research Institute Award for project CG005. Romain Ferrali acknowledges financial support from the French National Research Agency Grant ANR-17-EURE-0020, and the Excellence Initiative of Aix-Marseille University - A*MIDEX.

†Division of Social Science, New York University Abu Dhabi; Center for Behavioral Institutional Design (C-BID), New York University Abu Dhabi. Email: olivier.bochet@nyu.edu.

‡Aix-Marseille University, CNRS, AMSE, France. Email: romain.ferrali@univ-amu.fr.

§Monash University, Australia, and CEPR. E-mail: yves.zenou@monash.edu.

1 Introduction

Motivation Situations in which agents choose their interaction partners are common. Through these choices, agents form communities that endogenously generate payoffs for their members. Thus, communities can be viewed as groups of agents who share similar interests; i.e., similar payoffs. For example, individuals in the same Facebook group form a community because they share common interests and tend to exhibit similar characteristics. Likewise, individuals who belong to the same political party, working group, or interest group also form communities. Community members may also exhibit different, usually complementary characteristics, yet share similar interests. For instance, researchers often collaborate in teams, with different members contributing uniquely to various aspects of a paper. While no co-author can necessarily complete the paper alone, the combination of skills from all co-authors makes the paper possible. All co-authors share similar interests: once a paper is accepted by a journal, all authors typically receive the same return from that publication. In other words, it is sensible to assume that the payoffs of community members are homogeneous. Additionally, a paper may begin with some co-authors and later require new members with different skills from the existing ones. While there may be several agents with the same required skillset, not all are reachable, either directly by the initiator or by their recruited agents. Individual connections matter and limit the set of potential co-authors. Thus, the community formation process is typically sequential and constrained by a pre-existing network structure. Given this structure, recruitment is achieved through a *delegated* sequence of recruited agents making strategic offers to their neighbors.

Our approach is applicable to various examples (some of which we explore in detail in Section5), such as political activism, technology adoption, and criminal gang formation. Our aim is to enhance the understanding of the strategic community formation process within a network. We abstract from some strategic features that become relevant once the community formation is complete, such as strategic actions taken by members post-formation.[1] While integrating community formation into a game with ex-post action choices is also of interest, we assume that community payoffs already encompass these aspects. This allows us to provide a straightforward framework for studying various features relevant to the strategic formation of communities. Our approach also underscores the importance of community peers—a connected subgraph of the network—and their interactions with neighbors who are not community members.

Main Contribution We develop a model to operationalize this motivation. A seed (or initiator) forms a community on an exogenously given network. The links in the network are undirected and represent connections between agents. The community formation process is modeled as a sequential game, with recruitment constrained by the network structure. Given this structure, recruitment is achieved through a delegated sequence of recruited agents making strategic offers to their neighbors. This delegation means that the seed typically delegates part of the recruitment process to some of its recruited agents, who in turn delegate to their recruited agents, and so on, within the constraints of the given network structure. To emphasize

[1] See, e.g., Yariv and Baccara (2013) for a model where agents contribute to public projects after their group is formed.

the importance of connections, the initiator s can only recruit agents directly connected to her. Similarly, her recruited agents can only recruit agents directly connected to them, and so forth. Thus, community formation involves the seed delegating the recruitment process to subsequent layers of agents. In the sequential game we introduce, the seed is randomly chosen by Nature. In each subsequent period, a link between a community member and a community neighbor is randomly drawn, without replacement. The community member then makes a binary choice: whether to offer the neighbor to join the community. In turn, the neighbor makes the irreversible choice of whether to accept the offer and join the community. Starting from the seed, this process is sequential and finite due to the no-replacement assumption. We consider infinitely patient, forward-looking agents who make strategic offers to their neighbors, anticipating the equilibrium community that maximizes their payoffs.

Our first result (Theorem 1) characterizes the set of subgame perfect equilibria (SPE) of the game for arbitrary, homogeneous payoffs. We show that SPE are essentially unique, as each SPE maximizes the seed's payoff.[2] While Theorem 1 is important for understanding the community formation process, it does not specify the type and size of communities that may form at equilibrium. The only common characteristic of all SPE communities is that they maximize the seed's payoff. Additionally, we characterize Pareto-optimal equilibrium communities and highlight the conditions under which these equilibrium communities are efficient (i.e., welfare-maximizing).

An important by-product of Theorem 1 is that the assumption of homogeneous payoffs eliminates any tension on the seed's payoff that may arise from delegated recruitment. In this sense, delegation is always seed-optimal in a world with homogeneous payoffs. We next explore the robustness of our results when considering heterogeneous payoffs. We demonstrate that various conditions on preferences can restore our key insight from Theorem 1; specifically, that equilibrium communities maximize the seed's payoff. These conditions emphasize the role of bridges within the equilibrium communities, which must belong to the same preference class as the seed. Furthermore, we show that allowing for transfers can restore the seed-optimality of the delegation of recruitment. However, several examples illustrate that without assumptions on the network structure or payoffs, delegation can unravel and fail to deliver an equilibrium that maximizes the seed's payoff—see, for instance, Example 2.

We then impose more structure on agents' payoffs, assuming they depend on the number of individuals in the community (i.e., *community size*) and the number of individuals who are neighbors of the community (i.e., *community neighborhood size*). This added structure on payoffs allows us to capture a wide variety of economic situations within a tractable framework. We differentiate among three cases. First, we consider a case in which the payoff of each agent is increasing with community size but decreasing with community neighborhood size (case 1, *Increasing-Decreasing (ID) monotone* payoffs). An illustration of this case is political activism in an autocracy (Chwe, 2000; Siegel, 2009). The higher the number of activists (the community), the higher the payoff of being an activist. In contrast, the more witnesses (neighbors) there are, the lower is the payoff, as witnesses may report activists to the autocrat, thereby weakening

[2]Essential uniqueness solely relates to the seed's payoff maximization. For a given seed, different communities (and of different sizes) may emerge across equilibria.

the movement. Second, we study *Increasing-Increasing (II) monotone* payoffs (case 2), so that payoffs are increasing in both community size and community neighborhood size. Technology adoption (Chuang and Schechter, 2015; Breza, 2016) is a good illustration of this case. There are complementarities in technology adoption between adopters (the community) but also positive spillovers to non-adopters (the neighbors). Finally, we examine *Decreasing-Increasing (DI) monotone* payoffs (case 3), where payoffs decrease with the community size but increase with the community neighborhood size. This may be illustrated by criminal gangs (Carrington, 2011; Lindquist and Zenou, 2019), for which the community is the set of criminal gang members, while its neighborhood is the set of victims. Gang members are better off with larger numbers of (potential) victims, but compete for resources. As such, larger gangs decrease the payoff of individual gang members.

Our second main result (Theorem 2) further characterizes equilibrium communities in each of our three cases and, most importantly, allows us to rank these cases by the size of their equilibrium communities. First, we show that when payoffs are ID-monotone (case 1), equilibrium communities always encompass all network agents, resulting in a single community with no neighbors. Since payoffs increase with community size and decrease with neighborhood size, the seed is incentivized to recruit every agent in the network. Next, when payoffs are II-monotonous (case 2), equilibrium communities are always *dominating* communities; that is, communities where the union of community members and their neighbors includes all network agents. Because payoffs increase with both community size and neighborhood size, the seed has an incentive to ensure that every network agent is either a community member or a neighbor. Finally, when payoffs are DI-monotonous (case 3), equilibrium communities are always *exposed* communities; that is, communities such that no smaller community has a weakly larger neighborhood. Since payoffs decrease with community size and increase with neighborhood size, the seed prefers small communities with a large set of neighbors (i.e., exposed communities). Together, these three results allow us to rank our cases by the size of their equilibrium communities: the community size clearly decreases from case 1 to case 3.

Finally, we address a series of policy issues. First, we study the *key player* problem. As is clear from Theorem 2, II-monotonous payoffs (case 2) leave the most room for further investigation as the size of equilibrium communities is not entirely pinned down, yet remains tractable. For II-monotonous payoffs, we identify the key players in the network, defined as those players who contribute the most to the payoffs of equilibrium communities. Key players are assessed by how much equilibrium payoffs decrease once they are removed from the network. When comparing two agents, we show in Proposition 3 that for II-monotone payoffs, key players are partially characterized by two important network statistics resulting from removing an individual: the community size and the domination number.[3] In other words, for II-monotone payoffs, key players are significant because they act as gateways to some nodes—i.e., their removal results in smaller community sizes—or because they facilitate the creation of smaller dominating communities, allowing faster access to such nodes.

In the last part of the paper, we examine the impact of increasing network density (i.e.,

[3]A dominating community is a community such that the union of its members and its neighborhood covers the entire graph. The domination number is the size of the smallest dominating community in a graph.

adding links) on equilibrium outcomes. We first show that for any class of monotonic payoffs discussed previously, adding a link weakly increases the equilibrium payoff of the agents belonging to the equilibrium community. The addition of a link has two potential effects: it either makes existing communities more desirable or creates new communities that are potentially more desirable. An important observation is that the increase in payoffs does not necessarily raise overall welfare. The additional link may shrink the equilibrium community, potentially leading to an overall decrease in welfare. We then return to II-monotone payoffs (case 2). In Proposition 4, we show that additional links strictly increase the seed's equilibrium payoff (and thus, the payoff of all members of the equilibrium community) if and only if (i) the domination number strictly decreases, and (ii) payoffs are such that agents prefer the smaller minimum dominating community provided by this additional link. We conclude by providing three necessary and sufficient conditions (Proposition 5) under which adding a link to an existing network strictly reduces its domination number.

Related literature Our paper contributes to the games-on-network literature,[4] by examining the binary decision to join a community. The literature on network games has mostly focused on continuous actions (Jackson and Zenou, 2015). As in our model, there are some papers that have considered network games with binary actions (see, for example, Morris, 2000; Brock and Durlauf, 2001; Jackson and Yariv, 2005, 2007; Leister et al., 2022; Campbell et al., 2024). However, our model differs significantly as it focuses on binary actions involving whether or not to join a *community*, whereas the literature on games on networks typically addresses *individual* binary choices, such as adopting a new technology, a new operating system, or becoming politically active.

Our equilibrium characterization in terms of communities also relates to other network models that partition agents into endogenous community structures. These include risk sharing (Ambrus et al., 2014), interaction between market and community (Gagnon and Goyal, 2017), behavioral communities (Jackson and Storms, 2019), information resale and intermediation (Manea, 2021), technology adoption (Leister et al., 2022), and perceived competition (Bochet et al., 2021). This literature mainly focuses on the role of peers in the formation of *multiple communities* and characterizes the existence of multiple equilibrium communities within a network. Here, we focus on the formation of *one community* only and model not only the role of peers but also that of community neighbors who are *not* members of the community. In particular, we show that depending on whether community neighbors exert positive (cases 2 and 3) or negative (case 1) spillovers on community members, the characterization of the equilibrium communities can be very different (Theorem 2). This is one of the main novelties of our model. We believe it is important in many real-world situations. Consider, for example, the influential paper by Banerjee et al. (2013), who studied the role of peers and "key players" in the individual adoption of a microfinance program in India. They showed that both adopters and non-adopters have a key influence on the individual adoption of this microfinance program in a village. This corresponds to our case 2, in which the community is the set of adopters, while the community neighbors correspond to non-adopters who are exposed to the technology.

[4]For overviews, see Jackson (2008), Jackson and Zenou (2015), Bramoullé et al. (2016), and Jackson et al. (2017).

Because there are complementarities in adoption within the community and positive spillover effects of non-adopters on adopters, we show that the equilibrium community is a dominating community. That is, any person in the village is either an adopter or, if not, has a link to an adopter. Clearly, we can only obtain this result because the payoff function of each agent is a function of community members and their neighbors.

Finally, our model is related to the literature on community detection in computer science and physics.[5] Girvan and Newman (2002) were the first to develop an algorithm (the Girvan-Newman algorithm) to detect communities by progressively removing edges from the original network. The connected components of the remaining network were the communities. Since then, many algorithms have been developed to detect node communities (e.g., Newman and Girvan, 2004; Newman, 2006), overlapping communities, and link communities (e.g., Palla et al., 2005). In addition, statistical (Copic et al., 2009; Lancichinetti et al., 2011), information-theoretic (Rosvall and Bergstrom, 2007) and synchronization and dynamical clustering approaches (Yuan and Zhou, 2011) have also been developed to detect communities.

This literature takes a very different approach from ours, focusing purely on topology with no strategic behavior. In contrast, our model is primarily based on individual behavior and the subgame-perfect Nash equilibrium. While the network structure is important, in our model, the payoffs are key to determining which community emerges in equilibrium. (Theorem 2).

The rest of the paper is organized as follows. In the next section, we describe our model and introduce different notations. Section 3 provides a characterization of all SPE communities for arbitrary homogeneous payoffs while Section 4 considers heterogeneous payoffs. In Section 5, we consider homogeneous payoffs that are a function of community and neighborhood sizes. In Section 6, we study the policy implications of our model by examining the key-player policy and how adding links affects the equilibrium outcomes. Finally, Section 7 offers concluding remarks. The proof of all our results can be found in the Appendix.

2 Setting

Basic definitions A network (or graph) is a pair (N, G), where G is a network on the set of nodes (or agents) $N = \{1, ..., n\}$. For each pair $i, j \in N$, agents i and j are linked in G if and only if $ij \in G$. We assume that the network is *undirected*; that is, for each pair $i, j \in N$, $ij \in G \implies ji \in G$. A network (N, G) is *complete* if for each $i, j \in N$, $ij \in G$. We only consider networks (N, G) that are *connected*, i.e., for each $i, j \in N$, there exists a path $ii_0, i_0i_1, \ldots, i_kj$ of links between agents connecting agents i and j. In what follows, we simply refer to a network as G. For any given agent i, we say that agent j is a *neighbor* of i if $ij \in G$. Since G is undirected, agent i is also a neighbor of j.

Let $s \in N$ be called a *seed* (or initiator). A *community* C is a *connected subgraph* of G. Let $\mathcal{C}_s$ be the set of communities that include seed s. Likewise, let $\mathcal{C}$ be the set of all possible communities of G. Note that $\mathcal{C}_s \subseteq \mathcal{C}$ and that, for each possible initiator s, community $N \in \mathcal{C}_s$.

We now introduce the necessary ingredients of our strategic approach to community formation. Consider that time $t = 0, 1, ...$ runs discretely. Let $C^t \in \mathcal{C}$ be the community formed on

[5]For a recent overview of this literature, see Ahajjam and Badir (2022).

graph G at time t. We normalize $C^0 \equiv \emptyset$. Denote $P^t \subseteq G$ as the set of links that are *pending* at time t. No links are pending initially: $P^0 = \emptyset$. Finally, let $P_i(C^t) \equiv \{ij : ij \in G \text{ and } j \notin C^t\}$ be the set of i's links toward non-community members at time t.

Community-formation game Agents play a *game of community formation* on network G. At time $t = 0$, Nature randomly chooses a node $s \in N$ to be the *seed* according to some common-knowledge, full-support distribution. Seed s is offered to join community C^0. If the seed s rejects, the process ends. If it accepts, it becomes a community member: $C^1 = C^0 \cup \{s\}$. The links of seed s are then added to the set of pending offers. Hence, $P^1 = P^0 \cup P_s(C^1)$.

At each subsequent period $t \geq 1$, Nature randomly draws a link between community member $i \in C^t$ and non-community member j from the set of pending links P^t according to some common-knowledge, full-support distribution $\Pr(ij|P^t)$. The draw is made without replacement. Member i decides whether to offer to non-member j the possibility to join community C^t. If member i makes an offer and j accepts, she joins the community: $C^{t+1} = C^t \cup \{j\}$. Otherwise, the community remains unchanged: $C^{t+1} = C^t$.[6] Given that the process is without replacement, the link ij is then excluded from the set of pending links. In the event that j joins the community, j's links toward non-community members $P_j(C^{t+1})$ are added to the set of pending links. To ensure that all pending links are between community members and non-members, we also remove pre-existing pending links towards j. That is,

$$P^{t+1} = \begin{cases} P^t \setminus \{ij\} & \text{if } j \notin C^{t+1} \\ P^t \cup P_j(C^{t+1}) \setminus \big[\{ij\} \cup \{jk \in G : k \in C^t\}\big] & \text{otherwise.} \end{cases}$$

The process repeats until no additional offers can be made. Therefore, the process ends at the first period $\bar{t} \geq 1$ such that $P^{\bar{t}}$ is empty. When the process is over, the community $C \equiv C^{\bar{t}}$ is realized and payoffs accrue to each $i \in N$ according to the function $u_i : \mathcal{C} \to \mathbb{R}$. We normalize the payoffs of non-community members to zero and assume that payoffs are *homogeneous* among community members, as follows:

$$u_i(C) = \begin{cases} u(C) \in \mathbb{R} & \text{if } i \in C, \\ 0 & \text{otherwise} \end{cases}$$

To make the problem non-trivial, we assume that the seed never has an incentive to reject the offer from Nature and form an empty coalition.

Assumption 1 (Non-triviality)**.** *Let $\mathbb{G}$ be the set of connected graphs that can be formed with n nodes. For any $s \in N$ and any $G \in \mathbb{G}$, there is $C \in \mathcal{C}_s \neq \emptyset$, such that $u(C) > 0$.*

The process has a few properties that make it tractable. First, while all community members have the same realized payoff—which may be positive or negative— non-community members are all assigned a payoff of zero. Second, joining the community is *irreversible*: once an agent joins a community, she cannot leave it. Third, since links may only be drawn once from P^t,

[6] The seed continues to be selected to make offers until someone in the neighborhood of s accepts, if any. Since the link selection process is without replacement, the game ends if the seed has exhausted its set of link offers with rejections.

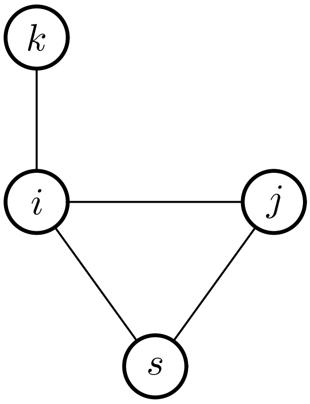


Figure 1: Running example with node s as the seed.

offers to join the community have *no recall.* In other words, an offer from i to j can only be made once. This feature ensures that the community-formation process is finite. Fourth, the assumption that offers are *unidirectional*—that is, they can only go from community members to community neighbors—ensures that the outcome of the community-formation process is a connected subgraph of G. Therefore, the outcome is necessarily a unique community. Finally, in this setting, agents are *perfectly forward-looking* and *infinitely patient.*

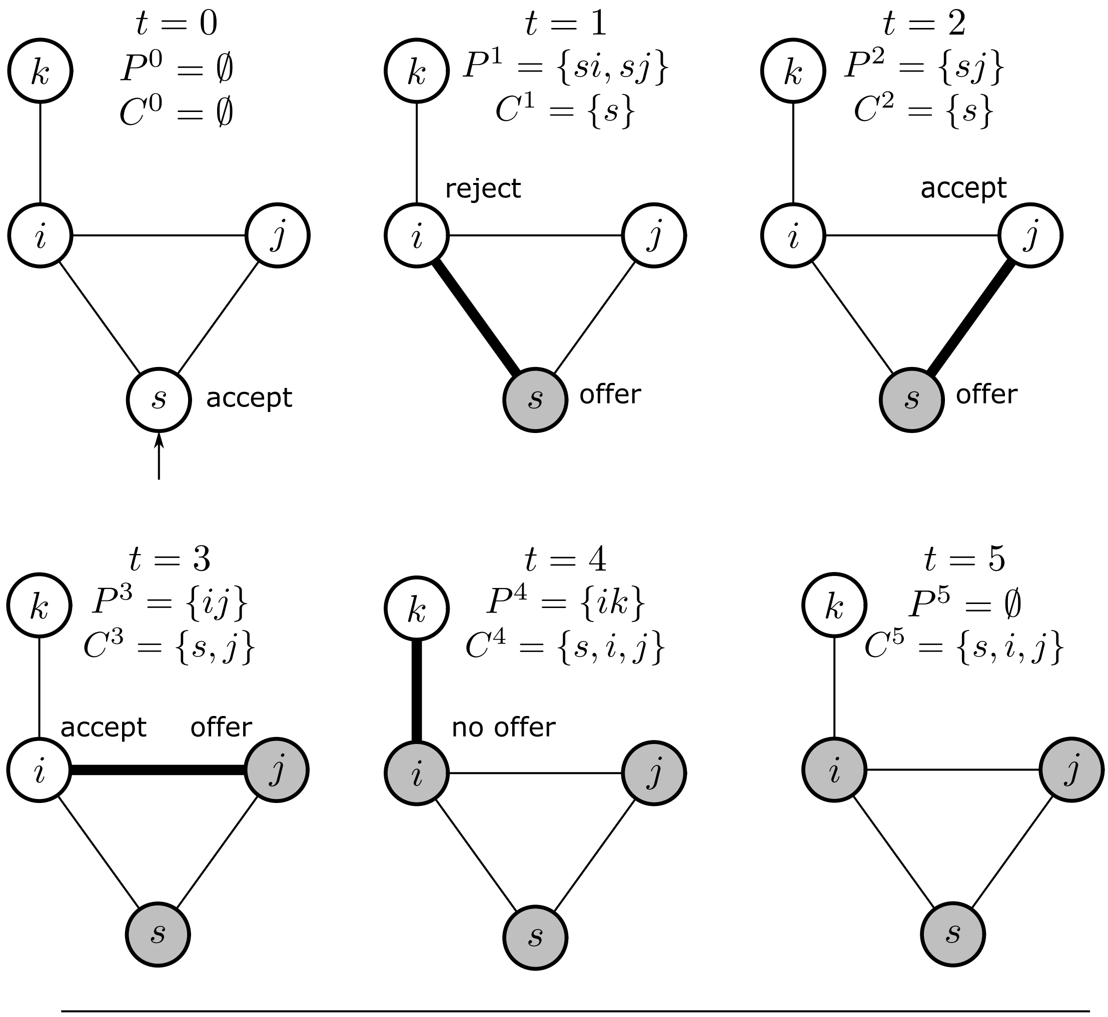
Our solution concept is *subgame perfect equilibrium (SPE).* In this context, a strategy profile $\sigma : H \to \{0, 1\}$ is a mapping from the set of histories H to $\{0, 1\}$, with 1 corresponding to making an offer/accepting it and 0 corresponding to not making an offer/rejecting it.

3 Equilibrium characterization

In this section, we will provide a characterization of the subgame perfect equilibria of the community formation game. Throughout the paper, we will use the network depicted in Figure 1 as our running example. In Figure 1, the seed s has already been chosen. We are at $t = 0$. Given that Assumption 1 holds, we know the seed will accept the role of initiator for recruiting potential community members.

Figure 2 illustrates the community-formation process for the network depicted in Figure 1 and an arbitrary payoff function. At the terminal history ($\bar{t} = 5$), the final outcome of the community-formation process is $C = \{s, i, j\}$. By the common payoff assumption, the realized payoff of each agent $\ell = s, i, j$ is $u_\ell(C) = 2$. Since player $k \notin C$, $u_k(C) = 0$. Figure 2 describes how the community-formation process leads to this outcome.

We show that for a given seed s, subgame perfect equilibrium (SPE) communities maximize the seed's payoff. In other words, the recruitment is always *seed-optimal.* This implies that SPE communities are *essentially unique* from the seed's point of view, even though they may differ in members and sizes. The intuition underlying the result is simple. Since community formation starts with seed s, any outcome of the community-formation game must be in $\mathcal{C}_s \subseteq \mathcal{C}$ the set of communities that include seed s. Since payoffs are homogeneous, if the seed finds community $C \in \mathcal{C}_s$ to be optimal, then its recruited members also find C to be optimal. As such, members of C, who are infinitely patient, may simply wait for the links that make up C



$u(\{s\}) = -1$ $\quad$ $u(\{s, j\}) = 1$ $\quad$ $u(\{s, i, k\}) = 1$

$u(\{s, i\}) = 0$ $\quad$ $u(\{s, i, j\}) = 2$ $\quad$ $u(\{s, i, j, k\}) = 2$

Figure 2: A community-formation process on our running example, with arbitrary payoffs. Thick links represent a link drawn from the set of pending links P^t. The process ends at $t = 5$ because $P^5 = \emptyset$. Its equilibrium outcome is community $C^5 = \{s, i, j\}$.

to be drawn, offer further members to join, and not offer non-members to join. In Appendix B, we fully characterize equilibrium profiles (Theorem B1).[7] We encapsulate the key properties of equilibria in the following theorem, which is a direct corollary of Theorem B1:

Theorem 1 (Equilibrium characterization)**.** *Suppose that Assumption 1 holds. SPEs exist and are seed-optimal. That is, given seed s, if the strategy profile σ is an SPE, then all of its equilibrium communities $C \in \mathcal{C}_s$ solve* $\max_{C \in \mathcal{C}_s} u_s(C)$.[8]

[7]Theorem B1 states that a profile is an equilibrium profile if and only if (i) the outcome of each subgame maximizes the seed's payoff within the set of communities that are feasible at that subgame's root history and (ii) the profile does not rely on non-credible rejections. In other words, if agent j could join a community C' such that $0 < u(C') < u(C)$, the equilibrium profile needs to ensure that j is not invited to join the community rather than relying on j rejecting an offer made to her.

[8]If Assumption 1 does not hold, it may be that the equilibrium community is empty. Remember that this result rests upon Theorem B1 in Appendix B.

In the network depicted in Figure 1, the set of possible communities with s as a seed is given by $\mathcal{C}_s = \{\{s\},\{s,i\},\{s,j\},\{s,i,j\},\{s,i,k\},\{s,i,j,k\}\}$. Theorem 1 implies that equilibrium communities must maximize the seed's payoff. The set of SPE communities is then a subset of $\mathcal{C}_s$. In Figure 2, equilibrium communities are either $\{s,i,j\}$ or $\{s,i,j,k\}$, since they both maximize the seed's payoff. Therefore, Theorem 1 allows for multiple equilibrium outcomes. Players may condition their actions on moves from Nature. Consider, for instance, a profile that has $\{s,i,j\}$ as an outcome if Nature draws link si at $t = 1$ and has $\{s,i,j,k\}$ as an outcome if Nature draws link sj at $t = 1$. This profile is an equilibrium profile, since these two outcomes are payoff-equivalent and maximize the seed's payoff.

Remark 1. *Theorem 1 implies that delegated recruitment entails no tension for the seed's payoff. In that sense, delegation is always seed-optimal in a world with homogeneous payoffs.*

Since all equilibrium communities maximize the seed's payoff, they all generate the same payoff $u(C)$, although the set of agents enjoying this payoff may differ across equilibria. In our working example (Figure 2), both equilibria $\{s,i,j\}$ and $\{s,i,j,k\}$ generate a payoff of 2. However, $u_k(\{s,i,j\}) = 0 < u_k(\{s,i,j,k\}) = u(\{s,i,j,k\}) = 2$. We formalize this notion of *essentially equal* communities as follows:

Definition 1. *Communities $C \neq C'$ are* essentially equal *if $u(C) = u(C')$.*

Based on Definition 1 and Theorem 1, we obtain the following result.

Corollary 1 (Uniqueness)**.** *Given seed s, any two SPE communities are essentially equal.*

Theorem 1 has the following important implication:

Remark 2. *Only community payoffs and the identity of the seed matters for finding equilibrium outcomes. There is an equilibrium profile that has community C as an outcome for any community $C \in \mathcal{C}_s$ that maximizes the seed's payoff. Therefore, the identity of the seed conditions the set of communities C_s that are sustainable in equilibrium. As such, the order in which links are drawn does not impact the equilibrium outcomes.*

Welfare considerations. Theorem 1 also has implications for welfare and efficiency. Note first that some (but not all) equilibrium communities are Pareto-optimal. We first recall the definition of Pareto optimality in our setting, then state the result.

Definition 2. Community $C \in \mathcal{C}$ *Pareto dominates* community C' if, for any $C' \in \mathcal{C}$, $u_i(C) \geq u_i(C')$ for each $i \in N$, and $u_j(C) > u_j(C')$ for some $j \in N$. Community C is *Pareto-optimal* if it is not Pareto dominated by any other community.

Among the equilibrium communities of seed s, only those that are not contained within another equilibrium community are Pareto-optimal. Since equilibrium communities are seed-optimal, the seed strictly prefers those to any other feasible community. Among equilibrium communities, those that admit another equilibrium community as a superset are dominated by their superset, since those additional members strictly prefer the superset. Formally:

Corollary 2 (Pareto optimality)**.** *Let $\mathcal{C}_s^*$ be the set of equilibrium communities associated with seed s. Pareto-optimal equilibrium communities are those equilibrium communities that do not admit another equilibrium community as a superset. Hence, $C \in \mathcal{C}_s^*$ is Pareto-optimal if there does not exist $C' \in \mathcal{C}_s^*$ such that $C \subset C'$.*

Going back to Figure 2, it is clear that $\{s, i, j, k\}$ is the unique Pareto optimal equilibrium community. We then discuss welfare. *Efficient* communities maximize welfare. Our notion of efficiency is local to the seed; in other words, we find the communities that maximize welfare among the communities that can be formed with seed s.

Definition 3. Community $C \in \mathcal{C}_s$ is *s-efficient* if, for any $C' \in \mathcal{C}_s$, we have $\sum_i u_i(C) \geq \sum_i u_i(C')$. With homogeneous payoffs, the welfare of community C simply writes $\sum_i u_i(C) = |C|u(C)$.

Since equilibrium communities are essentially equal, the largest equilibrium community is the most efficient one. Since only equilibrium communities are seed-optimal, larger, non-equilibrium communities must generate a lower payoff. Thus, the largest equilibrium community is s-efficient if the payoffs of larger communities drops sufficiently sharply to offset larger community size. Formally,

Corollary 3 (s-efficiency)**.** *Community $C^* \in \mathcal{C}_s^*$ is s-efficient if and only if it is the largest community in $\mathcal{C}_s^*$ and any larger community $C \in \mathcal{C}_s$ has $\frac{u(C)}{u(C^*)} \leq \frac{|C^*|}{|C|}$.*

4 Seed-optimality under heterogeneous payoffs

Instrumental to the results we have seen so far was the assumption that payoffs were homogeneous. Our central result is Theorem 1, which establishes that equilibrium communities are seed-optimal, thereby reducing the problem of community formation to a payoff-maximization problem over $\mathcal{C}_s$, the set of communities that seed s may form. Payoff homogeneity is key to delivering the result, as it ensures that delegating recruitment is seed-optimal (Remark 1): the members of a seed-optimal community will coordinate to realize said community, making offers to community members and making no offers to non-members.

In this section, we first illustrate the problem posed by heterogeneous preferences by considering two examples. Then, we provide a series of conditions that enforce seed-optimality under heterogeneous preferences. To represent heterogeneous preferences over the set of communities $\mathcal{C}$, we partition agents into $k \leq n$ preference classes. The preferences of agent $i \in N$ of preference class l are represented by the function $u_i : \mathcal{C} \to \mathbb{R}$ such that

$$u_i(C) = \begin{cases} v_l(C) & \text{if } i \in C, \\ 0 & \text{otherwise.} \end{cases}$$

To simplify matters, we assume that each agent has a unique favorite community for any seed. That is, we assume that $|\arg\max_{C \in C_s} u_i(C)| = 1$ for any $s, i \in N$ and denote C_i^P agent i's favorite community. We finally make an assumption that is analogous to, yet stronger than, Assumption 1:

Assumption 2. Agents have a strict preference towards joining communities; that is, $v_k(C) > 0$ for any k, C.

Assumption 2 is similar to Assumption 1 in that it transfers to a world of heterogeneous preferences the idea that at least one community is worth joining. At first sight, the assumption is stronger than Assumption 1 because it posits that *any* community is worth joining. Essentially, Assumption 2 simplifies matters by considering the problem of community formation from the point of view of community members only. In other words, we only consider instances in which community members may disagree upon which community to form and sidestep the issue of whether future community members would want to join such community. Relaxing Assumption 2 complicates the problem without providing further intuition. Indeed, a community C such that $u_i(C) < 0$ for some i is essentially not feasible without transfers. We may then redefine the problem by only considering the communities that are feasible both graphically and given a specific preference structure.

To get a sense of the difficulties that arise when considering heterogeneous preferences, we compare the following two examples. Our first example considers a star network with the seed as the hub:

Example 1. Star networks
Let G be a star network with s as the hub; that is, for each $i \neq s$, we have $si \in G$, and for each $i, j \neq s$, $ij \notin G$. Then equilibrium communities are seed-optimal. The intuition is simple: in a star network, the seed is a hub and retains full control over all offers. She can therefore ensure that the only offers that are made are to members of her favorite community. Since all agents have a strict preference towards joining a community, all offers are then accepted. The proof of this result is given in the Appendix. $\diamond$

Payoff heterogeneity is inconsequential in a world in which the seed has full control over the offers made. However, when delegating recruitment is necessary, misaligned preferences can hinder seed-optimal recruitment, forcing the seed to compromise. Since control over recruitment is crucial under payoff heterogeneity, seed-optimality can break down even in simple network structures, as we illustrate below

Example 2. A tension with efficiency and seed-optimality
Consider the line network $G = \{si, ij\}$. Suppose that $C_s^P = \{s, i\}$ and $C_i^P = \{s, i, j\}$. Community C_s^P cannot be an equilibrium community. Equilibrium communities are:

- $\{s\}$ if $u_s(\{s\}) \geq u_s(\{s, i, j\})$,
- $\{s, i, j\}$ if $u_s(\{s, i, j\}) \geq u_s(\{s\})$.

In this example, the seed and agent i disagree over the recruitment of agent j—the seed opposes it. Yet, upon receiving and accepting an offer from the seed, agent i has exclusive control over the recruitment of agent j. As such, s cannot prevent i from recruiting j and recruitment unravels. At equilibrium, recruitment and the formation of a community may still be beneficial for the seed. However, delegation unravels as the seed no longer obtains a payoff that aligns with the formation of her preferred community due to the misaligned interests of the seed and her directly recruited agent Note that the lack of seed-optimality would hold even if s could punish deviations from i, for example by recruiting some node k. To see why, consider the augmented network $G = \{si, ij, sk\}$ and suppose that $u_i(\{s, i, j, k\}) < u_i(\{s, i\})$. For s to threaten i with recruiting k, it must be that link sk is drawn after link ij. In other words, whether the threat is credible depends on the order in which links are drawn. $\diamond$

Comparing examples 1 and 2 highlights the relationship between seed-optimality, delegated recruitment, and payoff homogeneity. As discussed in the preceding section, delegating recruitment has no impact on seed-optimality under homogeneous payoffs. In contrast, recruitment unravels upon introducing payoff heterogeneity. Delegation becomes problematic, and seed-optimality breaks down due to the seed's lack of control over future recruitment.

There are, however, a variety of ways to restore seed optimality under heterogeneous preferences, albeit partially. These solutions broadly belong to two avenues. In the remainder of the section, we first describe these solutions then discuss them.

The first avenue consists in putting restrictions on preference heterogeneity. We provide below two different sets of conditions that may restore seed-optimality. The first set of conditions blends conditions on preferences with graphical considerations, identifying a joint condition on payoffs and network structure. In a nutshell, the conditions require that the seed has sufficiently many members that share her preference class and that such members are placed in favorable locations on the graph, from the seed's point of view. Formally,

Proposition 1. *Suppose that Assumption 2 holds and that seed s belongs to preference class k. If C_s^P is such that (a) for any $j \neq s \in C_s^P$, there is a path to s within C_s^P such that all nodes $i \neq j$ on this path belong to preference class k, and (b) any $k \in C_s^P$ that does not belong to preference class k has $N_k \setminus C_s^P = \emptyset$, then there is a seed-optimal equilibrium.*

Condition (a) ensures that each member of C_s^P is recruited, while condition (b) ensures that non-members of C_s^P are not recruited. The proposition extends the notion of seed control over recruitment by delegating recruitment to other agents of the same preference class and preventing members that belong to other preference classes from accessing non-members. Importantly, condition (a) implies that agents who do not share the seed's preferences cannot be cut vertices within C_s^P, echoing familiar results about the importance of bridges in diffusion processes (see e.g., Centola, 2018, 2021).

Another set of conditions on preferences *may* restore seed-optimality. These conditions do not rely on restrictions on the network structure, as they simply require that all agents prefer one set of communities over another. In this case, the equilibrium community will be one such preferred community. Formally,

Proposition 2. *Suppose that there is $\mathcal{C}_s^* \subset \mathcal{C}_s$ such that for any $C^* \in \mathcal{C}_s^*, \bar{C} \notin \mathcal{C}_s^*$, we have $v_k(C^*) > v_k(\bar{C})$ for any preference class k. If C is an SPE outcome, then $C \in \mathcal{C}_s^*$.*

Note that Proposition 2 does not necessarily imply seed-optimality. Nonetheless, it implies that under moderate heterogeneity, the set of equilibrium communities will be somewhat similar to the set of communities that are preferred by the seed. Importantly, this result will allow our next main result (Theorem 2 in Section 5) and its corollaries to survive under heterogeneous preferences.

The second avenue for preserving seed-optimality under heterogeneous preferences consists in introducing transfers. In Appendix C, we consider two extensions to our one-shot game with heterogeneous preferences. The first one extends the model to allow for side-payments and transfers that may be contracted upon. In other words, when agent i recruits agent j into the community, she also offers to j a contract that specifies the share of the community's value that j will transfer back to i after payoffs accrue. Contracts are enforced sequentially, with the last recruited agent being the first to enforce her contract. We show (Proposition C6) that contractible transfers restore seed-optimality, as each recruiting agent pockets all the surplus from her recruits, making all the surplus accrue to the seed. Our second extension considers informal contracts in a repeated game setting. We prove a result (Proposition C7) that resembles the folk theorem and broadly echoes the intuition gleaned from Proposition C6: if players are sufficiently patient, then a welfare-maximizing community may be enforced in equilibrium.

Overall, the results of this section suggest that seed-optimality may be restored when the preferences of community members are sufficiently similar, or when transfers allow compensating members to entice them to join the seed's favorite community. Communities whose members share interests that are too dissimilar will fail to form, as highlighted by Example 2.

5 Payoffs as a function of community and neighborhood sizes

Let us now return to the assumption of homogeneous payoffs from Section 2. Theorem 1 provides important insights into equilibrium communities and recruitment. However, as we have seen, many different communities can coexist at equilibrium. The common element among them is their optimality in terms of the seed's payoff, reducing the characterization of subgame perfect equilibrium (SPE) outcomes to a simple maximization problem over $\mathcal{C}_s$. Beyond this, we cannot provide much more detail about equilibrium communities without further assumptions on the payoff functions. We now introduce additional restrictions on payoffs to offer further insights into equilibrium outcomes and, more importantly, to encompass many situations that are salient in the network literature.

Let us introduce a specific payoff function under which payoffs vary according to the size of the community and that of its neighborhood, thereby capturing some of the externalities and spillovers exerted both by community members and some non-members (those directly connected to community members). That is, given $C \in \mathcal{C}$ and its associated neighborhood $N_C \equiv \{i \notin C : \exists ij \in G \text{ such that } j \in C\}$, we assume that the payoff generated by community C is given by

$$u(C) = v(|C|, |N_C|)$$

with $v : N^2 \to \mathbb{R}$ and where $|C|$ and $|N_C|$ denote the cardinal of the sets C and N_C. The set of remaining nodes, $A_C \equiv N \setminus \{C \cup N_C\}$, is the set of *anonymous* nodes.

We consider three cases.[9] For each of them, we provide a real-world application that has

[9]We do not study the fourth case in which payoffs decrease in both community and neighborhood sizes.

been studied in the network literature.

Case 1 (ID-monotonicity). $v(.)$ **is increasing in** $|C|$ **and decreasing in** $|N_C|$**.**

- **Political activism in an autocracy.** The community is the set of activists who benefit from having a larger cause (increasing in $|C|$). The neighborhood is a set of witnesses who may report activists to the autocrat and crush the movement (decreasing in $|N_C|$). Examples of political activism with network effects include Chwe (2000) and Siegel (2009).

Case 2 (II-monotonicity). $v(.)$ **is increasing in both** $|C|$ **and** $|N_C|$**.**

- **Technology adoption**. The community is the set of adopters, while the neighborhood represents non-adopters that are exposed to the technology. There are complementarities in adoption and spillover effects on the non-adopters. While not adopting, exposed non-adopters also modify their production technology in ways that complement that of adopters. Examples of technology adoption with network and spillover effects include Conley and Udry (2001, 2010), Bandiera and Rasul (2006), and Leister et al. (2022).[10]

Case 3 (DI-monotonicity). $v(.)$ **is decreasing in** $|C|$ **and increasing in** $|N_C|$**.**

- **Criminal gangs.** The community is the set of (criminal) gang members, while its neighborhood is the set of gang victims. Gang members are better off when the number of victims increases (i.e., utility increases in $|N_C|$) but are worse off when there is more competition for resources. As such, larger gangs decrease the payoff of any individual gang member (i.e., utility decreases in $|C|$). Examples of (criminal) gang networks include Calvó-Armengol and Zenou (2004), Baccara and Bar-Isaac (2008), Herings et al. (2009), Ballester et al. (2010), Mastrobuoni and Patacchini (2012), Mastrobuoni (2015), and Herings et al. (2021).[11]

The additional structure on payoffs afforded by function $v(.)$ allows for a strengthening of the equilibrium characterization. By complementing Theorem 1, our key result is that all equilibria can be ranked across all three cases. Overall, gang-formation-type problems, where v is DI monotone , involve fewer community members than technology-adoption-type problems, where v is II monotone; that is, v is monotone in both its arguments. In turn, II-monotonicity generates fewer equilibrium community members than activism-type problems in which v is ID monotone. In the remainder of this section, we introduce a few concepts, then state our main result, and finally present its underlying economic intuition.

We define a series of specific communities that play an important role in the subsequent results. The first one is the notion of a *dominating community*, a community such that the union of community members and its neighborhood covers the entire graph. This definition relates to the standard graph-theoretic concept of a *dominating set* (König et al., 2014). While a dominating set $D \subseteq N$ is a subset of N such that $D \cup \{j : ij \in G, i \in D, j \notin D\} = N$, a dominating community has the additional requirement that D is a connected subgraph of G.

Indeed, instances in which agents neither want to grow the community nor its neighborhood seem to be at odds with the core idea of community formation.

[10]For overviews, see Chuang and Schechter (2015) and Breza (2016).

[11]For overviews, see Carrington (2011) and Lindquist and Zenou (2019).

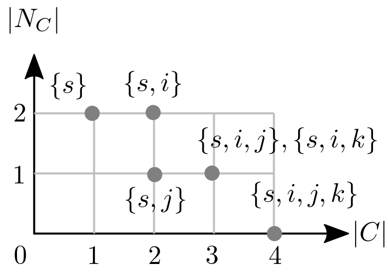


Figure 3: All communities in $\mathcal{C}_s$ for our running example (Figure 1). Points represent sets of communities.

Definition 4 (Dominating communities)**.** Given graph G, let $\mathcal{C}^D \equiv \{C \in \mathcal{C} : A_C = \emptyset\}$ be the set of *dominating communities*. Likewise, let $\mathcal{C}^{D,\min} = \{C \in \mathcal{C}^D : |C| = \min_{C' \in \mathcal{C}^D} |C'|\}$ be the set of *minimal dominating communities*, and $d \equiv |C|$ for $C \in \mathcal{C}^{D,\min}$ be the *domination number* of G. We can index the definitions with s to define the same notions for the communities whose seed is s; $\mathcal{C}_s^D$, $\mathcal{C}_s^{D,\min}$, and d_s.[12]

We now introduce the additional concept of an *exposed community*, a community in which no smaller community has a weakly larger neighborhood. Formally,

Definition 5 (Exposed communities)**.** Given graph G, let $\mathcal{C}^E \equiv \{C \in \mathcal{C} : |C'| < |C| \Rightarrow |N_{C'}| < |N_C|\}$ be the set of *exposed communities*. Likewise, let $\mathcal{C}^{E,\max} = \{C \in \mathcal{C}^E : |C| = \max_{C' \in \mathcal{C}^E} |C'|\}$ be the set of *maximal exposed communities*, and $\tilde{d} \equiv |C|$ for $C \in \mathcal{C}^{E,\max}$ be the *exposition number* of graph G. We can index the definitions with s to define the same notions for the communities whose seed is s; $\mathcal{C}_s^E$, $\mathcal{C}_s^{E,\max}$, and $\tilde{d}_s$.[13]

We provide two illustrations of these two concepts. We first revisit the example of Figure 1 and next look at an arbitrary graph. Figure 3 represents all communities in $\mathcal{C}_s$ obtained from Figure 1 as a function of their size $|C|$ and the size of their neighborhood $|N_C|$. The x-axis is the size of a community, while the y-axis is the size of its neighborhood. Points represent sets of communities. For instance, point $(3, 1)$ represents both communities $\{s, i, j\}$ and $\{s, i, k\}$.

Inspecting the graph of Figure 1, it is easy to see that the set of *dominating communities* of seed s is $\mathcal{C}_s^D = \{\{s,i\},\{s,i,j\},\{s,i,k\},\{s,i,j,k\}\}$. The smallest such community is $\{s, i\}$, implying that the set of minimum dominating communities of seed s is $\mathcal{C}_s^{D,\min} = \{\{s, i\}\}$. As such, the domination number of seed s is $d_s = 2$.

Figure 3 also helps identify *exposed communities*. Recall that exposed communities are communities in which no smaller community has a weakly larger neighborhood. Community $C = \{s, j\}$ is not exposed, since community $C' = \{s\}$ is smaller and has a weakly larger neighborhood. $N_C = \{i\}$ and $N_{C'} = \{i, j\}$, which implies that $|N_C| = 1 < 2 = |N_{C'}|$. More generally, in Figure 3, a point that admits another one at its upper left cannot be exposed. Conversely, exposed communities are points that have no points at their upper left. As such,

[12]Hence, $\mathcal{C}_s^D \equiv \{C_s \in \mathcal{C}_s : A_{C_s} = \emptyset\}$, $\mathcal{C}_s^{D,\min} \equiv \{C_s \in \mathcal{C}_s^D : |C_s| = \min_{C'_s \in \mathcal{C}_s^D} |C'_s|\}$, and $d_s \equiv |C_s|$ for $C_s \in \mathcal{C}_s^{D,\min}$.

[13]Hence, $\mathcal{C}_s^E \equiv \{C_s \in \mathcal{C}_s : |C'_s| < |C_s| \Rightarrow |N_{C'_s}| < |N_{C_s}|\}$, $\mathcal{C}_s^{E,\max} = \{C_s \in \mathcal{C}_s^E : |C_s| = \max_{C'_s \in \mathcal{C}_s^E} |C'_s|\}$, and $\tilde{d}_s \equiv |C_s|$ for $C_s \in \mathcal{C}_s^{E,\max}$.

the set of exposed communities for seed s is $\mathcal{C}_s^E = \{\{s\}\}$. Trivially, the largest of such community is $\{s\}$, and $\mathcal{C}_s^{E,\max} = \{\{s\}\}$. Therefore, the exposition number of seed s is $\tilde{d}_s = 1$.

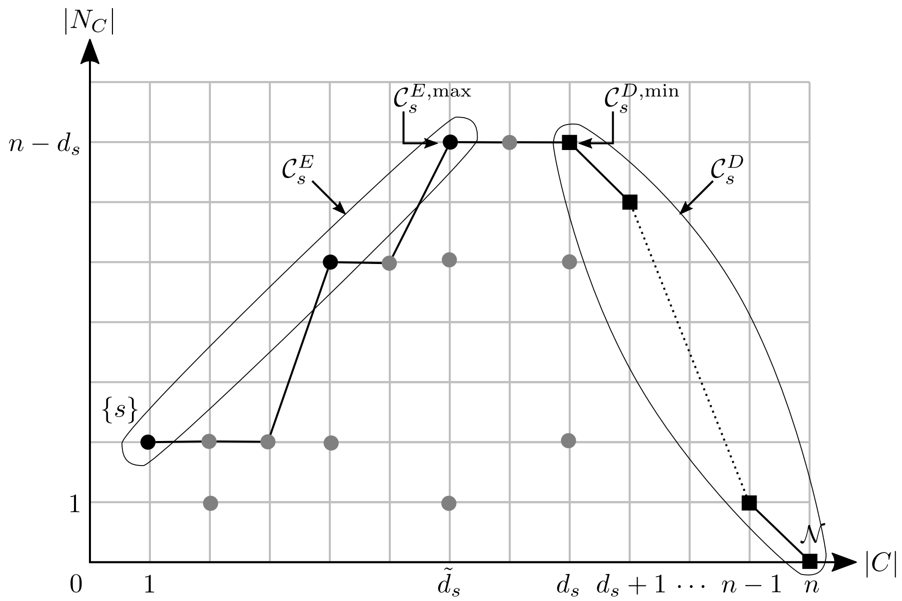


Figure 4: **Communities $\mathcal{C}_s$ of an arbitrary seed s on an arbitrary graph.** Points are sets of essentially equal communities. Black circles are exposed communities. Black squares are dominating communities. We omit communities whose size ranges from $d_s + 1$ to $n - 1$. The black line joins communities that have the largest neighborhood $|N_C|$ for a given size $|C|$.

In Figure 4, we illustrate these concepts in more detail, using an arbitrary graph. Note that $\{s\}$ is the only community of size 1. From community $\{s\}$, one can form communities of any size up to the complete community N. As such, Figure 4 has points for all $|C_s| \in \{1, \ldots, n\}$. The black line joins communities that have the largest neighborhood $|N_{C_s}|$ for a given size $|C_s|$. Recall that d_s is the domination number of seed s (i.e., the size of its smallest dominating community). The following lemma formally shows that this line is non-decreasing from $|C_s| = 1$ to $|C_s| = d_s$ and then decreasing from $|C_s| = d_s$ to $|C_s| = n$.

Lemma 1. *Consider seed s, and let $n_s^*(k) = \max_{\{C \in \mathcal{C}_s : |C| = k\}} |N_C|$. It must be that n_s^* is non-decreasing on $\{1, \ldots, d_s\}$ and decreasing on $\{d_s, \ldots, n\}$.*

Figure 4 allows the identification of the set of dominating communities $\mathcal{C}_s^D$: all communities on the black line that are to the right of d_s are dominating communities; they are represented as black squares in Figure 4. The smallest such communities form the set of minimum dominating communities $\mathcal{C}_s^{D,\min}$.

Figure 4 also allows the identification of the set of exposed communities $\mathcal{C}_s^E$. It is easy to see that no community to the right of d_s is exposed, since the minimum dominating community is smaller and has more neighbors. Considering the region of the graph to the left of d_s, it is also easy to see that communities under the black line are not exposed, as there is a community on the black line that has more neighbors and is weakly smaller. Similarly, communities that are on the black line and are not at a kink are not exposed. Indeed, they have a community to their left that has just as many neighbors. As such, the set of exposed communities $\mathcal{C}_s^E$ is the set of black circles on Figure 4. The largest such communities form the set of maximum

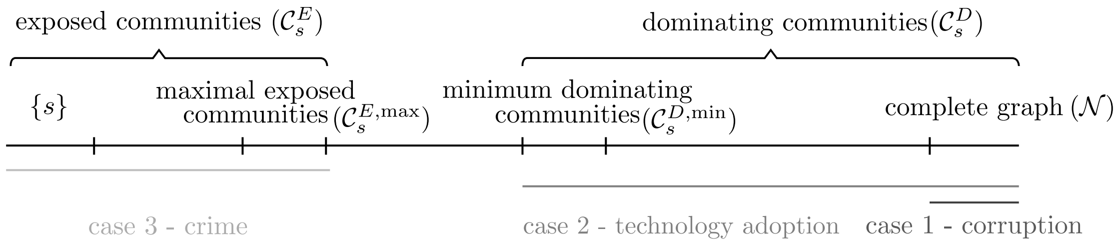


Figure 5: **Illustration of Theorem 2.** It may be that $\mathcal{C}_s^{E,\max} = \mathcal{C}_s^{D,\min}$.

exposed communities $\mathcal{C}_s^{E,\max}$. As shown in the figure, the exposition number must be smaller than the domination number: $\tilde{d}_s \leq d_s$. In our running example, $\tilde{d}_s = d_s$, while in this arbitrary example, $\tilde{d}_s < d_s$.

Recall that $\mathcal{C}_s^*$ is the set of equilibrium communities with seed s. We are now ready to state the main result of this section, one of the main results of the paper.

Theorem 2 (Equilibrium characterization)**.** *We have*

1. **Political activism.** *Suppose v is ID-monotone. Then, $\mathcal{C}_s^* = \{N\}$ for any seed s.*

2. **Technology adoption.** *Suppose v is II-monotone. Then, $\mathcal{C}_s^* \subseteq \mathcal{C}_s^D$ for any seed s.*

3. **Criminal gangs.** *Suppose v is DI-monotone. Then, $\mathcal{C}_s^* \subseteq \mathcal{C}_s^E$ for any seed s.*

Theorem 2 characterizes subgame-perfect equilibrium communities for each of our three cases. Figure 5 provides a graphical summary of what Theorem 2 actually pins down.

In the ID-monotone case (political activism, case 1), the unique equilibrium is obviously the complete community. Since the payoffs are increasing in community size and decreasing in neighborhood size, the seed has an incentive to hire every agent in the network.

In the II-monotone case (technology adoption, case 2), SPE are *dominating communities.* Indeed, by II-monotonicity, payoffs are increasing in both community and neighborhood sizes. The seed has an incentive to have every network agent to either be a community member or a neighbor. In other words, the seed incentives are to form a dominating community.

In the DI-monotone case (criminal gangs, case 3), SPE are *exposed communities.* To see this, consider a community C_s that is not exposed. There is then another community C'_s such that $|C'_s| < |C_s|$ and $|N_{C'_s}| \geq |N_{C_s}|$. By DI-monotonicity, payoffs are decreasing in the community size and increasing in the neighborhood size. Therefore, the seed prefers C'_s to C_s.

Remark 3. *Theorem 2 provides a full characterization only for the case in which v satisfies ID-monotonicity: the unique equilibrium community is the set of all agents. For the other two cases, our theorem does not provide a complete characterization; yet, it delivers bounds on the size of equilibrium communities owing to the notions of dominating (case 2) and exposed (case 3) communities. Note, however, that in both cases, there are generically many such dominating or exposed communities. Importantly, such communities are not essentially equal. Theorem 2 allows, however, narrowing down the set of equilibrium candidates to particular classes of communities. Moreover, by Theorem 1, the equilibrium community is necessarily the one that maximizes the seed's payoff.*

Despite these limitations, Theorem 2 has two important corollaries. First, it allows ranking equilibrium communities by size.

Corollary 4 (Ranking)**.** *Let C_s^{*k} be an equilibrium community associated with seed s when $v(.)$ satisfies either of cases $k = \{ID, II, DI\}$.*[14] *We have*

$$|C_s^{*ID}| \geq |C_s^{*II}| \geq |C_s^{*DI}|.$$

Considering again the example of Figure 1 and applying Theorem 2, we obtain the unique equilibrium community C_s^* for seed s:

1. If v is ID-monotone, then $C_s^* = \{s, i, j, k\}$.
2. If v is II-monotone, then $C_s^* \in \mathcal{C}_s^D = \{\{s, i\}, \{s, i, j\}, \{s, i, k\}, \{s, i, j, k\}\}$.
3. If v is DI-monotone, then $C_s^* \in \mathcal{C}_s^E = \{\{s\}, \{s, i\}\}$.

Theorem 2 also allows to identify the graph that is optimal for the seed. Recall that $\mathbb{G}$ denotes the set of connected graphs that can be formed with n nodes. Out of all networks $G \in \mathbb{G}$, we seek to identify the one that maximizes the seed's equilibrium payoff. Since there could be many such graphs, we require additionally that this graph minimizes the number of links.

Corollary 5 (Optimal graph)**.** *For any $G \in \mathbb{G}$, let $\mathcal{C}_{G,k}^*$ be the set of equilibrium communities for seed s for a value function in case $k \in \{ID, II, DI\}$. Let $\mathcal{G}_k^* \subseteq \mathbb{G}$ solve $\max_{G \in \mathbb{G}, C \in \mathcal{C}_{G,k}^*} u_s(C)$, and let $\mathcal{G}_k^{*,\min}$ be the set of networks in $\mathcal{G}_k^*$ that minimize the number of links. Consider a star network S with s the hub and the remaining $n-1$ nodes a spoke. We have that $S \in \mathcal{G}_k^{*,\min}$ for any k.*

It turns out that the star network is the optimal network for the seed in all three cases. This result is intuitive: the star network minimizes the number of links, and the seed can recruit all agents in the network. Furthermore, on a star network, all communities are both dominating and exposed, affording most flexibility to the seed in forming communities.

Remark 4. *Theorem 2 and Corollary 4 also hold when preferences are heterogeneous, that is, when the value function v_l satisfies the same case $k \in \{ID, II, DI\}$ for any preference class l. Indeed, monotonicity implies that complete, dominating, and exposed communities are preferred by all agents when preferences are ID, II, and DI-monotone respectively. As such, Proposition 2 implies that equilibria will be complete, dominating, and exposed communities respectively.*

6 Policy implications: Key players and denser networks

We now examine two important policy and targeting questions. First, we identify the *key players* in the network; that is, the players who contribute the most to the payoff of the equilibrium community.[15] Second, we examine the impact of increasing network density (i.e., adding links)

[14] For instance, case $k = ID$ refers to ID-monotonicity. We use similarly notations for the the two other cases.

[15] See Zenou (2016) for an overview of the literature on key players in the network.

on equilibrium outcomes. For each question, we state a series of general results, then examine in detail the case in which function v is II-monotone (i.e., case 2, technology adoption).[16]

6.1 Key players

For a given seed s, key players are the nodes that contribute the most to the payoff obtained by s, which is the the payoff of the equilibrium community (Theorem 1). In other words, key players are the nodes whose removal decreases the seed's equilibrium payoff the most in the network.

Let G^{-i} be the subgraph induced by removing i, and G_s^{-i} be the component of G^{-i} that includes s. In our running example (Figure 1), if we remove node i (and its links), we obtain G^{-i}, which has two separate components: k and $\{s, j\}$. Of these, only one component includes s; that is, $G_s^{-i} = \{s, j\}$.

Definition 6 (Key players)**.** Consider seed s on graph G, with equilibrium community C^*. For any $i \neq s$, let C_s^{-i*} be an equilibrium community on G_s^{-i}, and let $\Delta u_s^{-i} \equiv u(C_s^*) - u(C_s^{-i*})$ be the *contribution of* i *to seed* s in terms of payoffs. Node i is a *key player* if she has the highest contribution to seed s (i.e., if $\Delta u_s^{-i} \geq \Delta u_s^{-j}$ for any $j \neq s$). Let S_s be the set of such key players.

We partially identify key players when v is II-monotone. Interestingly, two statistics derived from the graph G_s^{-i} turn out to be crucial in determining key players: the size n_s^{-i} of G_s^{-i} and the domination number d_s^{-i} of G_s^{-i} (see Definition 4). In our running example where $G_s^{-i} = \{s, j\}$, $n_s^{-i} = 2$. Since the set of dominating communities is $C_s^{-i,D} = \{\{s\}, \{j\}\}$, the domination number is $d_s^{-i} = 1$.

Proposition 3 (Key players in case 2 (technology adoption))**.** *Consider seeds s and $i, j \neq s$. If $n_s^{-i} < (\leq) n_s^{-j}$ and $d_s^{-i} \geq d_s^{-j}$, then $\Delta u_s^{-i} > (\geq) \Delta u_s^{-j}$.*

This proposition shows that we can determine the key player between two agents by simply comparing the two statistics of G_s^{-i}, n_s^{-i} (size) and d_s^{-i} (domination number). This implies a *partial ordering* of players' contributions.

Let us now discuss and apply this proposition for Figure 1. Recall that there are three agents besides s. Figure 6 plots n_s^{-l} and d_s^{-l} for the nodes $l \in \{i, j, k\}$.

[16]We consider neither case 1 (political activism), because it is trivial, nor case 3 (criminal gangs), because no clear policy results emerge.

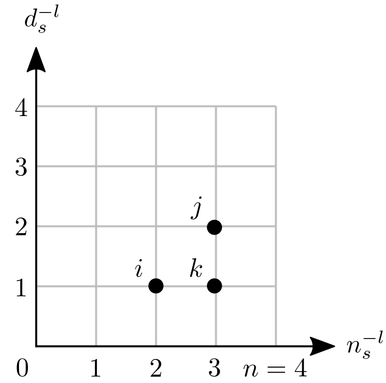


Figure 6: Key players for the network in Figure 1

Figure 6 illustrates why n_s^{-l}, the size of the remaining community, matters. Using Proposition 3, we obtain that $\Delta u_s^{-i} > \Delta u_s^{-k}$. We see that i and k have the same domination number $d_s^{-i} = d_s^{-k} = 1$. In contrast, agent i has a lower community size, since $n_s^{-i} = 2 < n_s^{-k} = 3$.[17] In other words, under II-monotonicity, removing i is more costly in terms of payoffs because it reduces more the size of the remaining community that s may form. As such, i is more "important" than k.

Figure 6 also illustrates why d_s^{-l}, the domination number of the remaining community, matters. Using Proposition 3, we obtain $\Delta u_s^{-j} \geq \Delta u_s^{-k}$. We see that j and k have the same community size $n_s^{-j} = n_s^{-k} = 3$[18] but agent j has a higher domination number, since $n_s^{-j} = 2 > n_s^{-k} = 1$. In other words, under II-monotonicity, if payoffs put more weight on neighbors than on community size, being able to form communities with large neighborhoods (i.e., small minimum dominating communities) is important. Removing j makes it more difficult to form a small minimum dominating community than removing k. Consequently, j is more "important" than k. Since this reasoning holds only when payoffs put more weight on neighbors than on community size, j is only weakly more important than k.

In summary, Proposition 3 helps narrow down the set of candidate key players, but it cannot fully characterize the set S_s. For example, in the network of Figure 1, it cannot compare agents i and j, since $n_s^{-i} < n_s^{-j}$ but $d_s^{-i} < d_s^{-j}$. However, since $\Delta u_s^{-i} > \Delta u_s^{-k}$, $k \notin S_s$, which implies that $S_s \subseteq A = \{i, j\}$. This logic generalizes into the following corollary, which gives a necessary condition for being a key player.

Corollary 6. *Fix seed s, and let v be II-monotone. If $i \in S_s$, then for all $j \neq i, s$, we have either $n_s^{-j} \geq n_s^{-i}$ or $d_s^{-j} < d_s^{-i}$.*

Corollary 6 has a simple graphical interpretation. Figure 7 generalizes Figure 6 to an arbitrary graph. Note that $n_s^{-i} \leq n - 1$ and $d_s^{-i} \leq n_s^{-i}$. As such, all nodes of the graph must be under the dotted line. Corollary 6 states that all nodes that have at least another node in their top-left quadrant (i.e., the grey points) are not in S_s. By eliminating these nodes, we are left with $S_s \subseteq A$.

[17] Observe that $G_s^{-k} = \{s, i, j\}$, which implies that $n_s^{-k} = 3$.

[18] Observe that $G_s^{-j} = \{s, i, k\}$, which implies that $n_s^{-j} = 3$ and, since the set of dominating communities is $C_s^{-j,D} = \{\{s, i\}, \{i, k\}\}$, $d_s^{-j} = 2$.

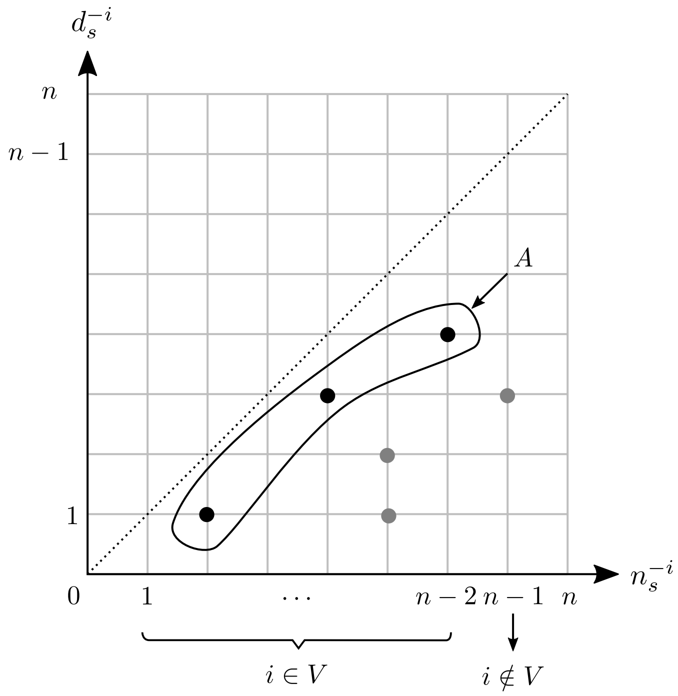


Figure 7: Graphical illustration of Corollary 6 for an arbitrary graph and an arbitrary seed s, with V being the set of cut vertices of that graph. Corollary 6 implies that $i \in S_s \Rightarrow i \in A$.

Let us formally define the well-known concept of a cut vertex (e.g., Bondy and Murty, 1976, p. 31) as it appropriately relates to the notion of key players:

Definition 7. A node i is a *cut vertex* of graph G if the induced subgraph G^{-i} is disconnected. Denote by V_G the set of cut vertices of graph G.

The notion of the cut vertex is important for our understanding of Corollary 6. The size of the remaining community n_s^{-i} is related to cut vertices, because while $n_s^{-i} \leq n-1$ for any node i, we have $n_s^{-i} < n-1 \iff i \in V_G$. Intuitively, when v is II-monotone, key players' influence may stem from two sources. First, they are gateways to some nodes (i.e., they are cut vertices since they have a small n_s^{-i}). Second, they are the fastest way to access such nodes, since they allow building small minimum dominating communities (i.e., high d_s^{-i}).

6.2 Network density

Let us now study the effect of increasing the network density on the equilibrium oucomes.

6.2.1 General case

We first cover a general payoff that encompasses all three cases. We examine the impact of adding a link to the network. Note that adding a link adds at most one neighbor to each existing community and can also create new communities.

To compare different graphs, we add the subscript G to all our previously defined variables. For instance, instead of $\mathcal{C}_s$, we use $\mathcal{C}_{G,s}$ to denote the set of feasible communities for seed s on graph G. Furthermore, we define $G' = G + ij$ as the graph that adds link ij to graph G_s.

Theorem 3 (Denser networks)**.** *Fix seed s and networks G and G'. Let C^*_G and $C^*_{G'}$ be equilibrium communities for seed s on G and G'. If v is ID-, II-, or DI-monotone, then, $u(C^*_{G'}) \geq u(C^*_G)$.*

Theorem 3 shows that, in all three cases, adding a link always weakly increases the equilibrium payoff of agents belonging to the equilibrium community. The additional link either makes existing communities more desirable or creates new communities that are potentially more desirable.

Theorem 3 implies neither that the agents who enjoy that payoff on the augmented graph G' are a subset of those who enjoyed the payoff on G, nor that this increased payoff increases welfare (defines as the total sum of utilities, see Definition 3). Even if adding a link increases the utility, it can shrink the equilibrium community, potentially leading to an overall decrease in welfare. Finally, while Theorem 3 guarantees that additional links do not decrease equilibrium payoffs, it is unclear which links strictly increase the equilibrium payoffs.

6.2.2 II-monotonicity (technology adoption, case 2)

To gain additional traction on the effect of variation in network density, we go back to case 2. We provide necessary and sufficient conditions which guarantee that additional links strictly increase the equilibrium payoffs.

Our first result (Proposition 4) is straightforward. Recall that when v is II-monotone, equilibrium communities are dominating (Theorem 2). Also note that since additional links facilitate creating communities, they cannot increase the domination number. Additional links strictly increase the seed's equilibrium payoff (and thus, the payoff of all members of the equilibrium community) if and only if the domination number strictly decreases and payoffs are such that agents prefer the smaller minimum dominating community provided by this additional link. To simplify matters, we assume that the valuation v of a community is singled-peaked as far as dominating communities are concerned. Formally,

Proposition 4. *Suppose that $v(k, n-k), k \in \{0, \ldots, n\}$ is single-peaked and reaches a maximum at k^*. Let C^*_G and $C^*_{G'}$ be equilibrium communities for seed s on the networks G and G', respectively. Then, $u(C^*_{G'}) > u(C^*_G)$ if and only if $d_{G',s} < d_{G,s}$ and $k^* < d_{G,s}$.*

Which links strictly reduce the domination number? Our next result (Proposition 5) provides the necessary and sufficient *graphical* conditions for the additional link ij to strictly reduce the domination number. Let us introduce them informally first. An additional link strictly reduces the domination number if and only if it satisfies one of the three conditions illustrated in Figure 8. First, the additional link *completes a community* (condition 1 in Proposition 5); that is, the link takes a community that was not dominating (in Figure 8, community $\{s, i\}$) and adds a neighbor to it (in Figure 8, node j), so the community becomes dominating. Second, it allows shrinking an existing dominating community by bypassing nodes whose sole function is to make a community connected (i.e., nodes that are cut vertices to the community), but do not bring any additional neighbors to the community. This second scenario admits two variants: one may either *bypass one cut vertex* (condition 2 in Proposition 5) or *bypass two cut vertices* (condition 3 in Proposition 5). In Figure 8, vertices k (condition 2) and k, l (condition 3) are cut vertices to the dominating communities $\{s, i, j, k\}$ (condition 2) and $\{s, i, j, k, l\}$ (condition 3). Additionally, these nodes do not bring neighbors to their communities. As such, the link ij allows bypassing them.

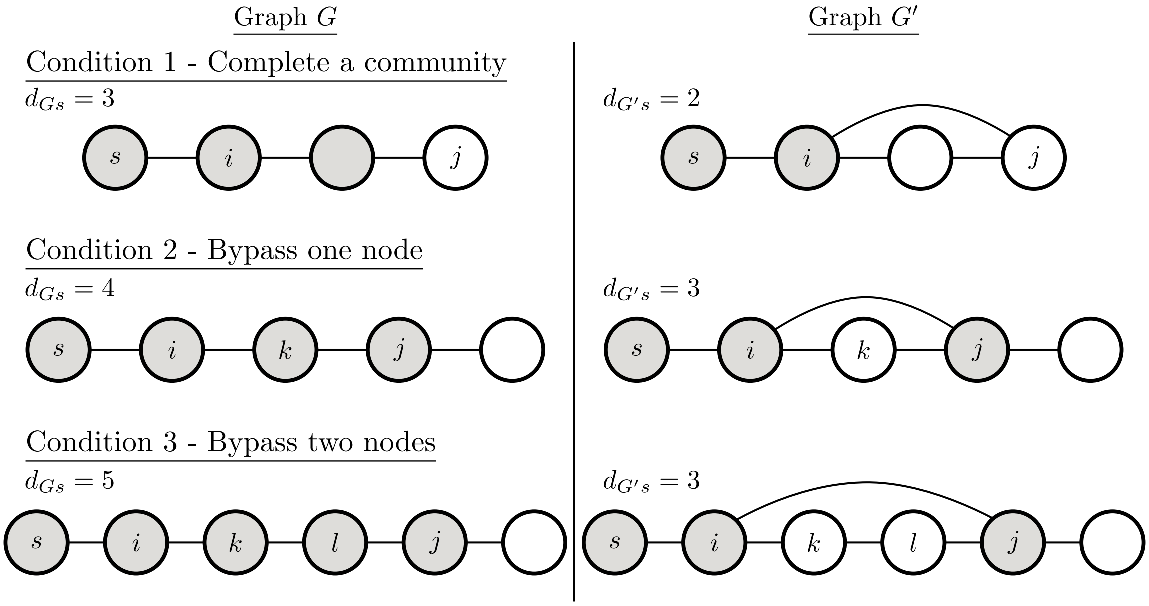


Figure 8: Illustration of Proposition 5. Gray nodes form a minimum dominating community for seed s.

While capturing condition 1 formally is relatively straighforward, stating conditions 2 and 3 formally requires additional notations. As illustrated in Figure 8, nodes that do not bring additional neighbors to a dominating community may be bypassed by an additional link. In other words, these nodes have no *private neighbors* in their community. Formally,

Definition 8 (Private neighbors)**.** Let $N_C^P(I)$ be the set of *private neighbors* of nodes $I \subseteq C$ for community C on graph G. Private neighbors are neighbors of I that are shared with no other members of C. That is,

$$N_C^P(I) = \{j : j \in N_C \text{ and there is } i \in I \subseteq C \text{ such that } ij \in G \text{ and } jk \notin G \text{ for any } k \in C \setminus I\}.$$

Consider our running example (Figure 1) and community $C = \{s, i, k\}$. We have $N_C^P(\{s\}) = N_C^P(\{i\}) = N_C^P(\{k\}) = \emptyset$, while $N_C^P(\{s, i\}) = \{j\}$. In other words, j is the private neighbor of the set $\{s, i\}$ for community C. Conditions 2 and 3 attempt to shrink a dominating community by bypassing some of its nodes. Only nodes that have no private neighbors may be bypassed.

Since nodes with no private neighbors do not add neighbors to the community, the sole reason for these nodes to be included in a minimum dominating community is that they make this community connected. In other words, these nodes are *cut vertices* to this community. Definition 7 introduces the notion of cut vertices of the whole graph G. We have a similar definition of cut vertices to a community. A vertex $i \in C$ is a *cut vertex* to community C if its removal makes C disconnected. We denote $\mathcal{V}_{G,C}$ as the set of cut vertices to community C on graph G and introduce the notion of *within-community paths*. That is, we define $\mathcal{P}_{G,C}(i, j)$ as the set of paths between nodes $i, j \in C$ such that all nodes on that path are in C. Node i is a cut vertex to C if and only if there are $j, k \in C$ such that $i \in p$ for any $p \in \mathcal{P}_{G,C}(i,j)$. In our running example (Figure 1), community $C = \{s, i, k\}$ has only one cut vertex: $\mathcal{V}_{G,C} = \{i\}$. We will see that if a dominating community C has cut vertices with no private neighbors, then these cut vertices may be bypassed by the addition of a link.

Another useful way to analyze whether a (sub)graph is connected is to examine its *block-cut tree*. The block-cut tree decomposes a community by separating it into a set of *blocks* (intuitively, components that do not contain cut vertices) tied to one another by cut vertices. Formally,

Definition 9 (Block-cut tree)**.** A *block* b_C of a community C on a graph G is a subgraph of C that is connected, has no cut vertices, and is maximal with respect to those properties. $B_G(C)$, the *block-cut* tree of community C on graph G, is a bipartite graph with bipartition $(\mathcal{B}_{G,C}, \mathcal{V}_{G,C})$, where $\mathcal{B}_{G,C}$ is the set of blocks of C on graph G and $V_{G,C}$ denotes the set of cut vertices of C. A block $b_C \in \mathcal{B}_{G,C}$ and a vertex $v_C \in \mathcal{V}_{G,C}$ are adjacent in $B_G(C)$ if and only if $v_C \in b_C$.

In our running example, consider the complete community $C = \{s, i, j, k\}$. This community has one cut vertex, $\mathcal{V}_{G,C} = \{i\}$, and has two blocks: $b_1 = \{s, i, j\}$ and $b_2 = \{i, k\}$. Its block cut tree is $B_G(C) = \{b_1 i, b_2 i\}$. We will see that links that reduce the domination number are those that make meaningful changes to the block-cut tree of G.

With this, we are now equipped to state our result formally:

Proposition 5. *Denote $G' = G + ij$ and consider seed $s \in N$. We have $d_{G',s} < d_{G,s}$ if and only if one of the following three conditions is met:*

1. **Complete a community.** *There is a community $C \in \mathcal{C}_{G,s}$ such that $|C| < d_{G,s}$, $i \in C$, and $A_C = \{j\}$.*
2. **Bypass one cut vertex.** *There is a community $C \in \mathcal{C}^D_{G,s}$ such that $|C| = d_{G,s}$, $i, j \in C$, i and j belong to distinct blocks of $B_G(C)$, there is a node $k \neq i, j, s$ such that $k \in \mathcal{V}_{G,C}$, $N_{G,C}(\{k\}) = \emptyset$, k has degree 2 on $B_G(C)$, and there is $p \in \mathcal{P}_{G,C}(i,j)$ such that $k \in p$.*
3. **Bypass two cut vertices.** *There is a community $C \in \mathcal{C}^D_{G,s}$ such that $|C| \leq d_{G,s} + 1$, $i, j \in C$, i and j belong to distinct blocks of $B_G(C)$, and there are two nodes $k, l \neq i, j, s$ such that $k, l \in \mathcal{V}_{G,C}$, $N_{G,C}(\{k, l\}) = \emptyset$, k, l both have degree 2 on $B_G(C)$, $\{k, l\} \in \mathcal{B}_{G,C}$, and there is $p \in \mathcal{P}_{G,C}(i,j)$ such that $k, l \in p$.*

Proposition 5 spells out the only three cases for which adding a link to a graph reduces the domination number. Adding a link to a graph both adds neighbors to existing communities and allows forming new communities. The additional link reduces the domination number if and only if it completes an existing community and makes it dominating, or allows forming a new, "better" dominating community. Condition 1 captures the former, while conditions 2 and 3 jointly capture the latter.

Condition 1 is relatively straightforward. The only way to complete a community and reduce the domination number is to consider a community that has only one anonymous neighbor j and connect it to a member i of this community. Our running example (Figure 1) with seed s

and $G' = G + sk$ illustrates this condition. The link sk allows completing community $C = \{s\}$. We have $|C| = 1 < d_{G,s} = 2$. The link sk satisfies $s \in C$ and $A_C = \{k\}$. By condition 1, we obtain $d_{G'_s} = 1 < d_{G,s}$.

We illustrate conditions 2 and 3 in the context of the simplified examples introduced in Figure 8. Consider condition 2 first. On graph G, community $C = \{s, i, j, k\}$ and link ij match condition 2. C is a minimum dominating community of seed s; as such, $C \in \mathcal{C}^D_{G,s}$ and $|C| = d_{G,s}$. Community C has cut vertices $\mathcal{V}_{G,C} = \{i, k\}$, blocks $\mathcal{B}_{G,C} = \{\{s, i\}, \{i, k\}, \{j, k\}\}$, and block-cut tree $B_G(C) = \{\{s, i\}i, \{i, k\}i, \{i, k\}k, \{j, k\}k\}$. Since $i \in \{s, i\}$ and $j \in \{j, k\}$, i and j belong to distinct blocks of $B_{G,C}$. Node k has no private neighbors in C; that is, $N_{G,C}(\{k\}) = \emptyset$. Furthermore, k has degree 2 on $B_G(C)$, as it is connected to blocks $\{i, k\}$ and $\{j, k\}$. Finally, the path $p = i, k, j$ is within C and has $k \in p$. By condition 2, $d_{G',s} = 2$, since community $C \setminus \{k\} \in \mathcal{C}^{D,\min}_{G',s}$. It is easy to check that condition 3 applies to the relevant graph in Figure 8.

We now discuss the necessity of each part of condition 2. Condition 2 shrinks a community by one node; as such, to strictly decrease $d_{G,s}$, the community C that is shrunk needs to be a minimum dominating community (i.e., a dominating community with $d_{G,s}$ members). The bypassed node k needs to be a cut vertex with no private neighbors. If k had private neighbors, she could not be removed. If k had no private neighbors and was not a cut vertex, then k could be made redundant in G, and so C would not be a minimum dominating community. For the link ij to bypass k, k must be on a path between i and j. This path connects i to j and is going through a series of blocks and cut vertices. However, i and j may not belong to the same block, for otherwise, there are at least two distinct paths between i and j: one with k on it, and the other without, meaning that k is already bypassed. Finally, for the link ij to guarantee that the resulting community $C \setminus \{k\}$ is connected, removing k must not prevent accessing other blocks. In other words, k must have degree 2 on the block-cut tree.

Condition 3 extends condition 2 to the removal of two nodes. Since condition 3 shrinks a community by two nodes, to strictly decrease $d_{G,s}$, the community C that is shrunk can be larger than that in condition 2. Specifically, it needs to be a dominating community with size at most $d_{G,s} + 1$. Similar to condition 2, the removed nodes k, l need to be cut vertices that jointly have no private neighbors. As in condition 2, the remaining conditions on the path between i and j and the block-cut tree ensure that the additional link ij actually bypasses nodes k, l without disconnecting $C \setminus \{k, l\}$.

To summarize, for II-monotone payoffs (technology adoption, case 2), we provide three alternative necessary conditions for an increase in the total welfare of the members of the equilibrium community. Each of these conditions reduces the domination number in the network obtained by adding a link (Proposition 5), which increases the total welfare of the equilibrium community (Proposition 4).

7 Conclusion

This paper aims to develop a game-theoretic framework to model the formation of a community and to understand how the community is affected by its members as well as its neighbors.

We have three main results. First, for arbitrary payoffs, there is essentially a unique subgame perfect equilibrium (SPE) that maximizes the payoff of the seed. In this sense, delegating part of the recruitment to other agents entails no loss to the seed/initiator. Second, by having payoffs depend on the size of the community and its neighbors, we show that three realistic cases emerge, corresponding to (i) political activism, (ii) technology adoption, and (iii) criminal gangs. The equilibrium community is complete in the first case, a dominating community in the second case, and an exposed community in the last case. This implies that we can rank the size of each community, with the largest in the first case and the smallest in the last case. Third, when comparing two agents in an equilibrium community, we can identify the key player using two sufficient statistics: the size of the remaining network and the domination number. We also provide conditions that ensure adding a link to a network increases the welfare of the equilibrium community.

When considering heterogeneous payoffs, we identify two distinct avenues for shaping real-life communities. The first avenue involves forming communities with relatively similar preferences, reflecting the common observation that members of real-life communities often share similar interests. As preferences become more dissimilar, the resulting communities will diverge further from the preferences of their seed members. Communities may still align with the preferences of their founders if similarly-inclined "zealots" carefully control the recruiting process, a mechanism often seen in radical organizations such as terrorist groups or clandestine parties. The second avenue involves compensating members, either through contracts or repeated interactions—the latter being more realistic in the informal settings that typically characterize network formation. This approach to community formation can be observed in clientelistic parties or corruption networks, where newer members often "pay their dues" to their sponsors.

References

Ahajjam, S. and H. Badir (2022). Community detection in social networks. In: Biswas, A., Patgiri, R. and B. Biswas (Eds.), *Principles of Social Networking. The New Horizon and Emerging Challenges,* Singapore: Springer, 91–107.

Ambrus, A., M. Mobius, and A. Szeidl (2014). Consumption risk-sharing in social networks. *American Economic Review 104*(1), 149–82.

Baccara, M. and H. Bar-Isaac (2008). How to organize crime. *Review of Economic Studies 75*(4), 1039–1067.

Ballester, C., A. Calvó-Armengol, and Y. Zenou (2010). Delinquent networks. *Journal of the European Economic Association 8*, 34–61.

Bandiera, O. and I. Rasul (2006). Social networks and technology adoption in Northern Mozambique. *Economic Journal 116*, 869–902.

Banerjee, A., A. Chandrasekhar, E. Duflo, and M. Jackson (2013). The diffusion of microfinance. *Science 341*(1), 6144.

Bochet, O., M. Faure, Y. Long, and Y. Zenou (2021). Perceived competition in networks. Unpublished manuscript, Monash University.

Bondy, J. and U. Murty (1976). *Graph Theory with Applications.* New York: North Holland.

Bramoullé, Y., B. Rogers, and A. Galeotti (2016). *The Oxford Handbook of the Economics of Networks.* Oxford: Oxford University Press.

Breza, E. (2016). Field experiments, social networks, and development. In: Y. Bramoullé and B.W. Rogers and A. Galeotti (Eds.), *The Oxford Handbook of the Economics of Networks, Vol. 4,* Oxford: Oxford University Press.

Brock, W. and S. Durlauf (2001). Discrete choice with social interactions. *Review of Economic Studies 68*, 235–260.

Calvó-Armengol, A. and Y. Zenou (2004). Social networks and crime decisions: The role of social structure in facilitating delinquent behavior. *International Economic Review 45*, 939–958.

Campbell, A., D. J. Thornton, and Y. Zenou (2024). Strategic diffusion: Public goods vs. public bads. CEPR Discussion Paper.

Carrington, P. J. (2011). Crime and social network analysis. In: J. Scott and P.J. Carrington (Eds.), *The Sage Handbook of Social Network Analysis,* London: SAGE Publications, 236–255.

Centola, D. (2018). *How Behavior Spreads: The Science of Complex Contagions.* Princeton: Princeton University Press.

Centola, D. (2021). *Change: How to Make Big Things Happen.* New York: John Murray Press.

Chuang, Y. and L. Schechter (2015). Social networks in developing countries. *Annual Review of Resource Economics 7*, 451–472.

Chwe, M. S.-Y. (2000). Communication and coordination in social networks. *Review of Economic Studies 67*(1), 1–16.

Conley, T. and C. Udry (2001). Social learning through networks: The adoption of new agricultural technologies in Ghana. *American Journal of Agricultural Economics*, 668–673.

Conley, T. and C. Udry (2010). Learning about a new technology: Pineapple in Ghana. *American Economic Review 100*, 35–69.

Copic, J., M. Jackson, and A. Kirman (2009). Identifying community structures from network data via maximum likelihood methods. *The B.E. Journal of Theoretical Economics 9*(1), Article 30.

Ferrali, R. (2020). Partners in crime? Corruption as a criminal network. *Games and Economic Behavior 124*, 319–353.

Gagnon, J. and S. Goyal (2017). Networks, markets, and inequality. *American Economic Review 107*(1), 1–30.

Girvan, M. and M. Newman (2002). Community structure in social and biological networks. *Proceedings of the National Academy of Sciences 99*, 7821–7826.

Herings, P. J.-J., A. Mauleon, and V. Vannetelbosch (2009). Farsightedly stable networks. *Games and Economic Behavior 67*, 526–541.

Herings, P. J.-J., A. Mauleon, and V. Vannetelbosch (2021). Horizon-k farsightedness in criminal networks. *Games 12*, 1–13.

Jackson, M. (2008). *Social and Economic Networks.* Princeton: Princeton University Press.

Jackson, M., B. Rogers, and Y. Zenou (2017). The economic consequences of social network structure. *Journal of Economic Literature 55*, 49–95.

Jackson, M. and L. Yariv (2005). Diffusion on social networks. *Economie Publique 16*(1), 69–82.

Jackson, M. and L. Yariv (2007). Diffusion of behavior and equilibrium properties in network games. *American Economic Review (Papers and Proceedings) 97*(2), 92–98.

Jackson, M. and Y. Zenou (2015). Games on networks. In P. Young and S. Zamir (Eds.), *Handbook of Game Theory, Vol. 4*, pp. 34–61. Amsterdam: Elsevier Publisher.

Jackson, M. O. and E. Storms (2019). Behavioral communities and the atomic structure of networks. Unpublished manuscript, Stanford University.

König, M. D., C. Tessone, and Y. Zenou (2014). Nestedness in networks: A theoretical model and some applications. *Theoretical Economics 9*, 695–752.

Lancichinetti, A., F. Radicchi, J. Ramasco, and S. Fortunato (2011). Finding statistically significant communities in networks. *PLOS ONE 6*, e18961.

Leister, M., Y. Zenou, and J. Zhou (2022). Social connectedness and local contagion. *Review of Economic Studies 89*(1), 372–410.

Lindquist, M. and Y. Zenou (2019). Crime and networks: Ten policy lessons. *Oxford Review of Economic Policy 35*(4), 746–771.

Manea, M. (2021). Bottleneck links, essential intermediaries, and competing paths of diffusion in networks. *Theoretical Economics 16*, 1017–1053.

Mastrobuoni, G. (2015). The value of connections: Evidence from the Italian-American mafia. *Economic Journal 125*(586), F256–F288.

Mastrobuoni, G. and E. Patacchini (2012). Organized crime networks: An application of network analysis techniques to the American mafia. *Review of Network Economics 11*(3), 1–43.

Morris, S. (2000). Contagion. *Review of Economic Studies 67*(1), 57–78.

Newman, M. (2006). Modularity and community structure in networks. *Proceedings of the National Academy of Sciences of the USA 103*, 8577–8582.

Newman, M. and M. Girvan (2004). Finding and evaluating community structure in networks. *Physical Review E 69*, 026113.

Palla, G., I. Derenyi, I. Farkas, and T. Vicsek (2005). Uncovering the overlapping community structure of complex networks in nature and society. *Nature 435*, 814–818.

Rosvall, M. and C. Bergstrom (2007). An information-theoretic framework for resolving community structure in complex networks. *Proceedings of the National Academy of Sciences of the USA 104*, 7327–7331.

Siegel, D. A. (2009). Social networks and collective action. *American Journal of Political Science 53*(1), 122–138.

Yariv, L. and M. Baccara (2013). Homophily in peer groups. *American Economic Journal: Microeconomics 5*, 69–96.

Yuan, W. and C. Zhou (2011). Interplay between structure and dynamics in adaptive complex networks: Emergence and amplification of modularity by adaptive dynamics. *Physical Review E 84*, 016116.

Zenou, Y. (2016). Key players. In: Y. Bramoullé, B.W. Rogers and A. Galeotti (Eds.), *The Oxford Handbook of the Economics of Networks,* Oxford: Oxford University Press, 244–274.

Appendices

A Proofs of results in the main text

Proof of Theorem 1. The result follows directly from Theorem B1 in Appendix B and Assumption 1. □

Proof of Corollary 1. Straightforward from the discussion in the text. □

Proof of Corollary 2. Note that Corollary 1 implies that $u(C) = u(C') \equiv \bar{u} > 0$.

Suppose that there is C' such that $C \subset C'$. Then for any $i \in C \cup C'$, we have $u_i(C) = u_i(C') = \bar{u}$. For any $i \notin C \cap C' = C'$, we have $u_i(C) = u_i(C') = 0$. For any $i \in C' \setminus C$, we have $u_i(C) = 0 < u_i(C') = \bar{u}$. So C' Pareto dominates C.

Suppose that there is no $C' \in \mathcal{C}_s^*$ such that $C \subset C'$. Note first that C is not Pareto-dominated by any $C'' \in \mathcal{C} \setminus \mathcal{C}_s$. Indeed, since $C'' \notin \mathcal{C}_s$, we have $s \notin C''$, implying $u_s(C'') = 0 < u_s(C)$. Note furthermore that C is not Pareto-dominated by any $C'' \in \mathcal{C}_s \setminus \mathcal{C}_s^*$. Indeed, Theorem 1 implies that $u_s(C'') < u_s(C)$. We finally show that C is not Pareto-dominated by any $C' \in \mathcal{C}_s^*$. Suppose that there is $C' \in \mathcal{C}_s^*$ with i such that $u_i(C') > u_i(C)$. This implies $i \in C', i \notin C$. Since $C \not\subset C'$, there must be j such that $j \in C, j \notin C'$. We then have $u_j(C) = \bar{u} > u_j(C') = 0$. So C' does not Pareto dominate C. So, C is Pareto optimal. □

Proof of Corollary 3. Let $\bar{u} \equiv u(C^*)$. Also note that with homogeneous payoffs, the welfare associated with community C, $\sum_{i \in N} u_i(C) = |C|u(C)$.

We first show necessity. Suppose that C^* is not the largest community in $\mathcal{C}_s^*$ and consider a larger community $C \in \mathcal{C}_s^*$. We have $|C|\bar{u} > |C^*|\bar{u}$. So C^* is not welfare maximizing.

Suppose now that there is a larger community C such that $\frac{u(C)}{u(C^*)} > \frac{|C^*|}{|C|}$. Rearranging obtains $|C|u(C) > |C^*|u(C^*)$. So C^* is not welfare maximizing.

We now show sufficiency. Suppose that C^* is the largest community in $\mathcal{C}_s^*$ and any larger community $C \in \mathcal{C}_s$ has $\frac{u(C)}{u(C^*)} \leq \frac{|C^*|}{|C|}$. Theorem 1 implies that $u(C^*) \geq u(C)$ for any $C \in \mathcal{C}_s$. So for any C that is weakly smaller than C^*, we have $|C^*|u(C) \geq |C|u(C)$. If C is larger than C^*, rearranging $\frac{u(C)}{u(C^*)} \leq \frac{|C^*|}{|C|}$ obtains $|C^*|u(C) \geq |C|u(C)$. So C^* is welfare maximizing. □

Proof of Example 1. Since G is a star, all offers must be from s to some other node $i \in N_s$, and each node i may receive at most one offer. Also note that in equilibrium, any on-path offer is accepted. To see why, let C and C' be the outcomes resulting from i's decision to accept and reject, respectively. Note that $i \in C$ and $i \notin C'$. So $u_i(C) > u_i(C') = 0$, implying that i accepts any offer made to her.

Fix σ^N and consider a profile σ^* such that s makes offers to any $i \in C_s^P$ and makes no offer to any $i \notin C_s^P$. Since all offers are accepted, this profile has C_s^P as an outcome. Since $u_s(C_s^P) > u_s(C)$ for any $C \neq C_s^P$, s has no incentive to deviate. So this profile is a SPE.

We now show that for the same σ^N, a profile σ that has $C \neq C_s^P$ as an outcome cannot be an equilibrium profile. Suppose σ is an equilibrium profile. Consider the path implied by σ^*. Follow the path from the terminal history up to the root and consider the first history h such that σ and σ^* differ. Since both σ and σ^* are equilibrium profiles, it must be that h is a history where s moves. Indeed, if h is a history where i moves, then i accepts under σ, as in under σ^*. All actions are binary. Label a^* and a the actions that s takes at h under σ^* and σ respectively. By construction, taking action a^* at h implies outcome C_s^P under σ. Taking action a implies some outcome $C \neq C_s^P$ under σ. Yet $u_s(C) < u_s(C_s^P)$, so s has a profitable deviation in taking action a^* instead of a under σ, implying that σ is not an equilibrium profile. □

Proof of Proposition 1. Let $I \equiv \{i \in C_s^P : u_i = u_s\}$ and $J \equiv C_s^P \setminus I$. Consider a profile σ^* such that:

1. If sender $i \in I$ is paired with receiver $j \in C_s^P$ at on-path history h such that C_s^P is feasible from h, then i makes the offer.

2. If sender $i \in I$ is paired with receiver $j \notin C_s^P$ at on-path history h such that C_s^P is feasible from h, i does not makes the offer.

3. Moves at any other history are specified using backward induction.

We show that σ^* is an equilibrium profile.

We first establish that at the last on-path history h such that receiver i moves, i must accept the offer. Since i moves as a receiver, condition (3) implies that her move is specified by backward induction. Suppose that i rejects at h under σ^* and denote C and C' the outcomes from rejecting and accepting, respectively. Note that $i \notin C$ and $i \in C'$. As such, i's payoff from rejecting is $u_i(C) = 0 < u_i(C')$, which is a contradiction. So i must accept at h.

We now establish that profile σ^* has C_s^P as an outcome. Conditions (1) and (a) imply that any $j \in C_s^P$ receives an offer from some $i \in I$. Conditions (2) and (b) imply that no $k \notin C_s^P$ receives an offer. Since all players accept their last on-path offer, C_s^P is the outcome.

We finally establish that players $i \in I$ have no incentive to deviate at the histories that are specified by conditions (1) and (2). Let C be the outcome from deviating. We have $u_i(C_s^P) \geq u_i(C)$, so i has no incentive to deviate. □

Proof of Proposition 2. Suppose not. That is, suppose that there is $C \notin \mathcal{C}_s^*$ such that there is an equilibrium profile σ that has C as an outcome. Follow the path from the root history and stop at the last history h_0 such that offerer i moves and some community $C^* \in \mathcal{C}_s^*$ is feasible from h_0.

Consider the subgame that starts at h_0 and denote h' the history that immediately follows a deviation from σ at h_0 and denote C' its outcome under σ. Notice that it must be that $C' \notin \mathcal{C}_s^*$ for otherwise i has a profitable deviation at h_0.

From h_0, construct a path σ' of offers and acceptances such that (a) C^* is the outcome, (b) no offers are made to any $j \notin C^*$, and (c) all offers are accepted.

Consider now the terminal history of σ' and move up σ' towards the root history h_0. Stop at the first history h_1 such that σ' and σ diverge. Consider player j_1 that moves at h_1 and suppose that the outcome under σ is C_1. Since σ is a SPE, it must be that $u_{j_1}(C_1) \geq u_{j_1}(C^*)$. We now establish that $j_1 \in C_1$. This is obvious if j_1 is an offerer at h_1. If j_1 is a recipient, she accepts under σ', by construction. Therefore, she rejects under σ. Since $C^* \in \mathcal{C}_s^*$, we have $u_{j_1}(C^*) > 0$, implying $u_{j_1}(C_1) > 0$, which then implies $j_1 \in C_1$. So we have $j_1 \in C_1 \cap C^*$ and $u_{j_1}(C_1) \geq u_{j_1}(C^*)$, implying that $C_1 \in \mathcal{C}_s^*$.

This argument may be iterated with a slight modification: from h_1, move up σ' towards the root history h_0. Stop at the second history h_2 such that σ' and σ diverge. Consider player j_2 that moves at h_2 and suppose that the outcome under σ is C_2. Notice that under σ, the outcome from deviating at h_2 is $C_1 \in \mathcal{C}_s^*$. Since σ is a SPE, it must be that $u_{j_2}(C_2) \geq u_{j_2}(C_1)$. We now establish that $j_2 \in C_2$. This is obvious if j_2 is an offerer at h_2. If j_2 is a recipient, she accepts under σ', by construction. Therefore, she rejects under σ. Since $C_1 \in \mathcal{C}_s^*$, we have $u_{j_2}(C_1) > 0$, implying $u_{j_2}(C_2) > 0$, which then implies $j_2 \in C_2$. So we have $j_2 \in C_2 \cap C_1$ and $u_{j_2}(C_2) \geq u_{j_2}(C_1)$, implying that $C_2 \in \mathcal{C}_s^*$.

We iterate this argument $k \geq 1$ times until reaching h_0. The argument implies that the outcome following a deviation from σ at h_0 is $C_k \in \mathcal{C}_s^*$. Yet, we have established previously that $C_k = C' \notin \mathcal{C}_s^*$. □

Proof of Lemma 1. We show that for any $k \in \{1, \ldots, d_s - 1\}$, we have $n_s^*(k) \leq n_s^*(k+1)$. Let $C_s^*(k)$ be a community that solves $\max_{C \in \mathcal{C}_s : |C| = k} |N_C|$. Since G is connected and $C_s^*(k)$ is not a dominating community, there is $i \in N_{C_s^*(k)}$ that has a neighbor $j \in A_{C_s^*(k)}$. So community $C_s^*(k) \cup \{i\}$ has at least $n_s^*(k)$ neighbors, and so $n_s^*(k) \leq n_s^*(k+1)$.

We now show that n_s^* is decreasing on $\{d_s, \ldots, n\}$. Note that for any $k \geq d_s$, it must be that $C^*(k) \in \mathcal{C}_s^D$. As such, $n_s^*(k) = n - k$, which is decreasing in k. □

Proof of Theorem 2. Note that by Theorem 1, an equilibrium community of seed s C_s^* satisfies $u(C_s^*) = \max_{C \in \mathcal{C}_s} u(C)$. Also note that by assumption 1, we have $C_s^* \neq \emptyset$.

Proof of case 1. Note that $N \in \mathcal{C}_s$ for any s, and that we have $|C| < n$ for any $C \neq N$ and $|N_C| \geq 0$ for any $C \neq N$. Therefore $\arg\max_{C \in \mathcal{C}_s} u_s(C) = \{N\}$.

Proof of case 2. We prove a useful lemma.

Lemma A2. *Consider graph G. If community $C \in \mathcal{C}_s \setminus \mathcal{C}_s^D$, then there is $C' \in \mathcal{C}_s^D$ such that $|C'| > |C|$ and $|N_{C'}| \geq |N_C|$.*

Proof of Lemma A2. Suppose $C \in \mathcal{C}_s \setminus \mathcal{C}_s^D$. Note that there is $C^* \in \mathcal{C}_s$ such that $|C^*| = |C| + 1$ and $|N_{C^*}| \geq |N_C|$. Indeed, if $C \in \mathcal{C}_s \setminus \mathcal{C}_s^D$, then there is $i \in N_C$ that has at least one neighbor $k \in A_C$, for otherwise C is a dominating community. So the community $C^* = C \cup \{i\}$ satisfies $|C^*| = |C| + 1 > |C|$, and $|N_{C^*}| \geq |N_C|$. Iterating this argument for all such nodes i, it must be that there is $C'' \in \mathcal{C}_s^D$ such that $|C''| > |C|$ and $|N_{C''}| \geq |N_C|$. □

For any $C \in \mathcal{C}_s \setminus \mathcal{C}_s^D$, Lemma A2 implies that there is $C^* \in \mathcal{C}_s^D$ such that $|C^*| > |C|$ and $|N_{C^*}| \geq |N_C|$. So $u(C^*) > u(C)$.

Proof of case 3. We prove the contrapositive. That is, we prove that if $C \notin \mathcal{C}_s^E$, then C is not an equilibrium community of seed s. If $C \notin \mathcal{C}_s^E$, then there is C' such that $|C'| < |C|$ and $|N_{C'}| \geq |N_C|$, implying that $u(C') > u(C)$. □

Proof of Corollary 4. By Theorem 2, it must be that $|C_s^{*1}| = n \geq |C_s^{*2}| \geq d_s$, and that $\tilde{d}_s \geq |C_s^{*3}|$. Since $d_s \geq \tilde{d}_s$, it must be that $|C_s^{*2}| \geq |C_s^{*3}|$. □

Proof of Corollary 5. Note first that the star graph S minimizes the number of links among all graphs $G \in \mathbb{G}$. As such, if $S \in \mathcal{G}_k^*$, then $S \in \mathcal{G}_k^{*,\min}$. To prove the corollary, it suffices to show that $S \in \mathcal{G}_k^*$ for any $k \in \{ID, II, DI\}$. In this proof, we subscript all objects with G to indicate the graph they belong to.

Consider $k = ID$. Theorem 2 implies that $\mathcal{C}_{G,k}^* = N$ for any $G \in \mathbb{G}$. Since the complete community N may be formed on any graph, we have that $\mathcal{G}_{ID}^* = \mathbb{G}$.

Consider $k = II$. Theorem 2 implies that $\mathcal{C}_{G,k}^* \subseteq \mathcal{C}_G^D$ for any $G \in \mathbb{G}$. Suppose that the value function v is maximized at $v(m, n-m)$. In other words, v is maximized by a dominating community C_m of size m. Such community may not be feasible on all graphs $G \in \mathbb{G}$. On S, any community is dominating, so $C_m \in \mathcal{C}_{S,II}^*$, implying that $S \in \mathcal{G}_{II}^*$.

Consider $k = DI$. Then v is maximized at $v(1, n-1)$. In other words, v is maximized by an exposed community C_1 of size 1. Such community may not be feasible on all graphs $G \in \mathbb{G}$. On S, any community is exposed, so $C_1 \in \mathcal{C}_{S,DI}^*$, implying that $S \in \mathcal{G}_{DI}^*$. □

Proof of Proposition 3. Consider $i, j \neq s$ such that $n_{s,-i} < (\leq) n_{s,-i}$ and $d_{s,-i} \geq d_{s,-j}$. Consider furthermore communities $C_i \in \mathcal{E}_{G_s^{-i}}, C_j \in \mathcal{E}_{G_s^{-j}}$. Note that if $C \in \mathcal{C}_{G_s^{-i}}^D$, then there is $C' \in \mathcal{C}_{G_s^{-j}}^D$ such that $|C'| = |C|$ and $|N_{G_s^{-j},C'}| > (\geq) |N_{G_s^{-i},C}|$. By P^2-monotonicity, it must be that $u(C') > (\geq) u(C)$. As such, we have $u(C_j) > (\geq) u(C_i)$, which implies $\delta_{is} > (\geq) \delta_{js}$. □

Proof of Corollary 6. Suppose not. That is, suppose there is $i \in S_s$ and j such that $n_s^{-j} < n_s^{-i}$ and $d_s^{-j} \geq d_s^{-i}$. Proposition 3 implies that $\Delta u_s^{-i} < \Delta u_s^{-j}$, which contradicts $i \in S_s$. □

Proof of Theorem 3. By Theorem 1, $C_{G_s}^*$ solves $\max_{C \in \mathcal{C}_{G_s}} u(C)$. As such, it suffices to show that $\max_{C \in \mathcal{C}_{G's}} u(C) \geq \max_{C \in \mathcal{C}_{G_s}} u(C)$.

Suppose we are in case 1. By Theorem 2, we have $C_{G_s}^* = C_{G's}^* = N$. As such, $\max_{C \in \mathcal{C}_{G's}} u(C) = \max_{C \in \mathcal{C}_{G_s}} u(C)$. Suppose now that v is increasing in $|N_C|$; that is, suppose that v matches cases 2 or 3 and note that any $C \in \mathcal{C}_G$ satisfies:

$$|N_{G'C}| = \begin{cases} |N_{GC}| + 1, & \text{if } i \in C \text{ and } j \in A_C \\ |N_{GC}| & \text{otherwise.} \end{cases}$$

Since v is increasing in $|N_C|$, then for any $C \in \mathcal{C}_{G_s}$, we have $u_{G'}(C) \geq u_G(C)$, which proves the claim. □

Proof of Proposition 4. Suppose that $d_{G',s} < d_{G,s}$ and $k^* < d_{G,s}$. By condition 2, it must be that $C_G^* \in \mathcal{C}_{G,s}^{D,\min}$. Proposition 5 implies that there is $C \in \mathcal{C}_{G',s}^D$ such that $|C| = d_{G',s} < d_{G,s}$. If $d_{G',s} \geq k^*$, then $u(C) > u(C_G^*)$. If $d_{G',s} < k^*$, then there is $C' \in \mathcal{C}_{G',s}^D$ such that $|C'| = k^*$. We have $u(C) > u(C_G^*)$.

Consider $s \in N$ such $d_{G',s} = d_{G,s}$. Then $u(C_{G'}^*) = u(C_G^*)$. Suppose now that $k^* \geq d_{G,s}$. Then it must be that C_G^* is essentially equal to $C_{G'}^*$, which implies $u(C_{G'}^*) = u(C_G^*)$. □

Proof of Proposition 5. We first show that if any of conditions 1, 2, 3 is met, then $d_{G',s} < d_{G,s}$. Suppose that condition 1 holds. Note first that by construction, $C \notin \mathcal{C}_{G,s}^D$ and $C \in \mathcal{C}_{G',s}^D$. Also note that $|N_{G'C}| = |N_{GC}| + 1$. Furthermore, note that it must be that $|C| = d_{G,s} - 1$. Indeed, suppose that $|C| < d_{G,s} - 1$. Since G is connected and $A_C = \{j\}$, it must be that there is $k \in N_{GC}$ such that $j \in N_{G,\{k\}}$. Therefore, we have that $C \cup \{k\} \in \mathcal{C}_G^D$, and $|C \cup \{k\}| = |C| + 1 < d_{G,s}$, a contradiction. So we have $|C| = d_{G,s} - 1 < d_{G,s}$, which proves the point.

Suppose now that condition 2 holds. Note that community $C' \equiv C\setminus\{k\}$ has $|C'| = d_{G,s}-1 < d_{G,s}$. To prove the point, it suffices to show that $C' \in \mathcal{C}^D_{G',s}$.

We first show that $C' \in \mathcal{C}_{G'_s}$. That is, we show that $\mathcal{P}_{G',C'}(x,s) \neq \emptyset$ for any $x \in C'$. If there is $p' \in \mathcal{P}_{G,C}(x,s)$ such that $k \notin p'$, then $p' \in \mathcal{P}_{G',C'}(x,s)$. Suppose now that there is no such p'. It must be that x, s belong to distinct blocks b_x, b_s respectively of C. Note that if there is a path between b_x, b_s on $B_{G'}(C')$, then $\mathcal{P}_{G,C'}(x,s) \neq \emptyset$. We show that such path exists. Let b_i, b_j be the blocks of i, j respectively. On $B_G(C)$, there is a path between b_x and b_s that goes through k. Since $k \in p$, there is also a path between b_i and b_j that goes through k. Since k has degree 2 on $B_G(C)$, it must also be that (without loss of generality) there is a path from b_x to b_i and from b_s to b_j such that k is on none of those two paths. Note that on $B_{G'}(C')$, the link ij creates a new block $b_{ij} = \{i,j\}$, and implies that $i, j \in \mathcal{V}_{G',C'}$. As such, the path $b_x, \ldots, b_i, i, b_{ij}, j, b_j, \ldots, b_s$ connects b_x to b_y.

We now show that $C' \in \mathcal{C}^D_{G',s}$. Since $N_{G,C}(\{k\}) = \emptyset$, it must be that $N_{G',C'} = N_{G,C} \cup \{k\}$. As such, $C' \in \mathcal{C}^D_{G',s}$.

Suppose now that condition 3 holds. Note that community $C' \equiv C \setminus \{k,l\}$ has $|C'| \leq d_{G,s} - 1 < d_{G,s}$. We prove the point as for condition 2. That is, we first show that $C' \in \mathcal{C}_{G',s}$, then that $C' \in \mathcal{C}^D_{G',s}$. The proof proceeds as for condition 2, but considers k, l instead of k. The additional requirement that $b_{kl} \equiv \{k,l\} \in \mathcal{B}_{G,C}$ implies that the path between $b_i(b_x)$ and $b_j(b_s)$ goes through k, b_{kl}, l instead of just k.

We now show that if $d_{G',s} < d_{G,s}$, then any of conditions 1, 2, 3 is met. Consider $C' \in \mathcal{C}^D_{G',s}$ and suppose that $d_{G',s} < d_{G,s}$. Suppose furthermore that $C' \in \mathcal{C}_{G,s}$. Then it must be that condition 1 is met for otherwise, either $C' \notin \mathcal{C}^D_{G',s}$ or $d_{G',s} = d_{G,s}$.

Suppose now that $C' \notin \mathcal{C}_{G,s}$. Then it must be that $i, j \in C'$ and $\mathcal{P}_{G',C'}(i,j) = \{\{i,j\}\}$ for otherwise, $C' \in \mathcal{C}_{G,s}$. We show that there must be $C \in \mathcal{C}^D_{G,s}$ that meets condition 2 or 3. To do so, we prove a useful lemma.

Lemma A3. *Suppose $G' = G + ij$. Consider seed s and community $C \in \mathcal{C}^D_{G',s}$ such that $i, j \in C$ and $\mathcal{P}_{G',C}(i,j) = \{i,j\}$. It must be that one of the following statements is true:*

1. *There is $k \in N_{G',C}$ such that $k \in N_{G',C}(b_1)$ and $k \in N_{G',C}(b_2)$ for $b_1 \neq b_2 \in \mathcal{B}_{G',C}$.*

2. *There are $k, l \in N_{G',C}$ such that there is a link between k and l and $k \in N_{G',C}(b_1)$ and $l \in N_{G',C}(b_2)$ for $b_1 \neq b_2 \in \mathcal{B}_{G',C}$.*

Proof of Lemma A3. Note that G is connected. In other words, there must be a path from i to j on G that does not go through the link ij. Suppose that both statements 1 and 2 are false. We show that this implies that G is disconnected. To do so, we show that on G', the only path from i to j is $p = i, j$. Since p is the only path from i to j with all nodes within C, if another path p' exists, it must go through nodes in $N_{G',C}$. Specifically, p' starts from i, then stays within C, then leaves C through some block b_1, and re-enter C through some other block b_2 and finally reach j. Yet, if statements 1 and 2 are false, p' cannot re-enter C through block b_2. □

Note that C' meets the requirements of Lemma A3. Suppose condition 1 of Lemma A3 holds. We show that $C = C' \cup \{k\}$ satisfies condition 2 of Proposition 5. By construction,

$C \in \mathcal{C}^D_{G,s}$. Furthermore, since $d_{Gs} > d_{G's}$, it must be that $d_{G,s} \geq d_{G,'s} + 1$. As such, $|C| = d_{G,s}$. Furthermore, since $C' \in \mathcal{C}^D_{G',s}$, it must be that $N_{G,C}(\{k\}) = \emptyset$. Additionally, on G, $k \in p$ for any $p \in \mathcal{P}_{G,C}(i,j)$. As such $k \in \mathcal{V}_{G,C}$ and i, j belong to different blocks of $B_G(C)$. It remains to show that k has degree 2 on $B_G(C)$. Suppose not. That is, suppose that there is a node m that belongs to some block $b_m \neq b_i, b_j$, with a link from b_m to k on $B_G(C)$. If that is so, then C' is not feasible on G'.

Suppose now that condition 2 of Lemma A3 holds. We show that $C = C' \cup \{k, l\}$ satisfies condition 3 of Proposition 5. By construction, $C \in \mathcal{C}^D_{G,s}$. Furthermore, $|C| = d_{G',s} + 2$. Since $d_{G',s} \leq d_{G,s} - 1$, we have $|C| \leq d_{G,s} + 1$. Furthermore, since $C' \in \mathcal{C}^D_{G',s}$, it must be that $N_{G,C}(\{k, l\}) = \emptyset$. Additionally, on G, $k, l \in p$ for any $p \in \mathcal{P}_{G,C}(i,j)$. As such $k, l \in \mathcal{V}_{GC}$ and i, j belong to different blocks of $B_G(C)$. We show as for condition 1 of Lemma A3 that k and l have degree 2 on $B_G(C)$. Finally, since $k, l \in \mathcal{V}_{G,C}$ and there is a link between k and l, then $\{k, l\} \in \mathcal{B}_{G,C}$. □

B Characterization of equilibrium profiles under homogeneous preferences

Recall that C^h is the community that is formed at history h and let $\mathcal{C}^h$ be the set of communities that are feasible from h.

Theorem B1. *The strategy profile σ is a SPE if and only if, for any subgame starting at history h at which sender i considers the offer ij, (i) its outcome C^*_h solves $\max_{C \in \mathcal{C}^h} u_i(C) \equiv \bar{u}_h$ if $\bar{u}_h \geq 0$ and $C^*_h = C^h$ if $\bar{u}_h < 0$, and (ii) if $j \notin C^*_h$ and $\max_{C \in \mathcal{C}^h : j \in C} u(C) > 0$, then i makes no offer to j under σ and, following a deviation from i, j accepts i's offer.*

Proof of Theorem B1. We show that a profile σ that satisfies conditions (i) and (ii) is a SPE. Consider the subgame that starts at history h. We first assume that $\bar{u}_h < 0$ and show that no player on path has a profitable one-shot deviation from the path implied by σ.

We first show that receivers on path have no profitable one-shot deviation from the path implied by σ. We begin by showing that σ implies that at any child history h_r of h such that a receiver moves, the receiver rejects the offer under σ. Consider a subgame that starts at a child history h_s of h such that a sender moves. Since $\mathcal{C}^{h_s} \subseteq \mathcal{C}^h$, it must be that $\bar{u}_{h_s} \leq \bar{u}_h < 0$. As such, condition (1) requires that the outcome of that subgame is C^{h_s}. For C^{h_s} to be the outcome, it must be that on the path from h_s, all receivers reject offers. That is, at any child history h_r of h such that a receiver moves, the receiver rejects the offer under σ.

We now focus on the path from history h implied by σ. Our argument implies that at any on-path history h_r such that receiver r moves, r rejects the offer under σ. We now show that r has no profitable one-shot deviation in accepting said offer. Since profile σ implies that all offers are rejected, it must be that the community that C^{h_r} that is formed at history h_r is C^h. Since all offers at any child history of h are rejected, the outcome of the subgame following acceptance is community $C^h \cup \{r\}$. Her payoff from accepting is then $u(C^h \cup \{r\}) \leq \bar{u}_h < 0$.

We then show that senders on path have no profitable one-shot deviation from the path implied by σ. Senders on path are indifferent between making an offer and not making it, since all offers are rejected. As such, they have no profitable deviation from the path implied by σ.

Together this implies that if $\bar{u}_h < 0$, then no player on path has a profitable one-shot deviation from the path implied by σ.

We now assume that $\bar{u}_h \geq 0$ and show that no player on path has a profitable one-shot deviation from σ.

Offerers have no incentive to deviate since their payoff from deviating is $u \leq \bar{u}_h$.

Consider now an on-path history h_r such that receiver r moves. Suppose first that r accepts on path and denote C the outcome from rejecting. She has no incentive to reject, as condition (i) implies that $\bar{u}_h \geq u_r(C) \geq 0$.

Suppose now that r rejects on path and denote C the outcome from accepting. Suppose furthermore that $r \in C^*_h$. This implies that there is a later on-path history such that r accepts (i.e., at h_r, r is delaying her entry). We show that $u(C) = u(C^*_h)$. Denote h_D the history following r's deviation notice that $C^*_h \in \mathcal{C}^{h_D}$, and (b) a sender must move at h_D. Since $C^*_h \in \mathcal{C}^{h_D}$, condition (i) implies that $\bar{u}_{h_D} = \bar{u}_h$; that is, $u(C) = u(C^*_h)$. As such, r is indifferent between accepting and rejecting. Suppose now that $r \notin C^*_h$. Note that it must be that $u(C) \leq 0$ for otherwise, σ would violate condition (ii). As such, r has no incentive to deviate.

We now show that a profile σ that does not satisfy condition (i) or (ii) is not a SPE. Suppose that σ admits a subgame starting at history h whose outcome community C does not satisfy (i). Consider the case where $\bar{u}_h < 0$. Since $\bar{u}_h < 0$, not satisfying (i) implies $C \supset C^h$. As such, there must be a receiver that accepts on path. This receiver has a profitable deviation in rejecting, since $u(C) \leq \bar{u}_h < 0$.

Consider now the case where $\bar{u}_h \geq 0$. Let C^* be a community that solves $\max_{C \in \mathcal{C}^h} u_i(C)$. Starting from history h, consider the last history h' on σ's path such that a sender j moves and $C^* \in \mathcal{C}^{h'}$ (note that it may be that $h' = h$). Since $C^* \in \mathcal{C}^{h'}$ and a sender moves, there must be a profile σ^* such that (a) C^* is the outcome of the subgame that starts at h', (b) Nature's moves are identical to those of σ, and (c) condition (ii) holds on the path from h' to C^*.

Consider now the path implied by σ^* from h' to its terminal history, which has outcome C^*. Start from the terminal history and stop at the first history h'' such that σ and σ^* differ and the outcome under σ is a community C'' such that $u(C'') < u(C^*)$. Note that it may be that $h' = h''$, in which case $C'' = C$. Let k be the player that moves at h''. By construction, h'' satisfies several properties:

- It is on the path implied by σ^* from h' to C^*.
- Player k chooses between two actions that have, under σ, outcomes C'' and $C^{*'}$ such that $u(C'') < u(C^*) = u(C^{*'})$ as an outcome.
- Player k chooses the action that has C'' as an outcome under σ.

We show that either player k has a one-shot profitable deviation, or σ^* does not satisfy condition (ii). Suppose first that k is a sender. As such, $k \in C'', C^{*'}$. Since $u_k(C^{*'}) = u(C^{*'}) > u(C'') = u_k(C'')$, k has a one-shot profitable deviation from σ. Indeed, she has an incentive to deviate to the action that has $C^{*'}$ as an outcome instead of C''.

Suppose now that k is a receiver. If k rejects under σ, she has a profitable deviation in accepting, as $u(C^{*'}) > 0$. Suppose now that k accepts under σ. If $u(C'') < 0$, then she has a profitable deviation in rejecting. If $u(C'') > 0$, then k has no profitable deviation, but σ^* must then violate condition (ii), a contradiction.

Consider now a subgame starting at history h at which sender s considers the offer sr such that (i) holds but (ii) does not. Notice that this requires that the outcome C_h^* of this subgame satisfies $u(C_h^*) = \bar{u}_h \geq \max_{C \in \mathcal{C}^h : j \in C} u(C) > 0$. Also notice that not satisfying condition (ii) implies that on path, it must be that s makes an offer to r and r rejects this offer. Since $r \notin C_h^*$, it must be that her on-path payoff is $u_r(C_h^*) = 0$. Let C be the outcome following a deviation from r (i.e., following r accepting the offer from s), and notice that $r \in C$. We establish that $u_r(C) > 0$. Suppose that the history following r's deviation is a terminal history. Then $|\{C \in \mathcal{C}_h : j \in C\}| = 1$, implying that $u_r(C) > 0$. Suppose that the history following r's deviation is not a terminal history. Then it is followed by a subgame beginning with a history h' such that some sender moves. Since σ satisfies condition (i), its outcome $C_{h'}^* = C$ solves $\max_{C \in \mathcal{C}^{h'}} u(C)$. Since $\mathcal{C}^{h'} = \{C \in \mathcal{C}_h : j \in C\}$, we have $u_r(C) > 0$. As such, r has a profitable deviation in accepting the offer from s, implying that σ is not a SPE. □

C Transfers

Community formation game with contracts. We extend the one-shot game described in Section 2 as follows. When i offers j to join the community, i also offers j a contract $t_{ji} : \mathbb{R}^+ \to \mathbb{R}^+$, such that $0 \leq t_{ji}(x) \leq x$. The contract specifies the transfer t_{ji} that j will make to i once the game is over, as a function of the budget available to j. Once the game is over with community $C \in \mathcal{C}_s$ as an outcome:

1. Payoffs accrue to each player i as per $u_i(C)$
2. Contracts are executed in the reverse order in which offers have been accepted.

Suppose that player $j \in C$ accepted an offer from $i \in C$, and suppose that j recruited the set of players $K \subset C$. The budget $b_j(C)$ available to j for transfers to i amounts to the utility that accrued to her and her incoming transfers; that is:

$$b_j(C) = u_j(C) + \sum_{k \in K} t_{kj}(b_k(C)),$$

and j's payoff writes

$$\omega_j(C) = \begin{cases} b_j(C) - t_{ji}(b_j(C)) & \text{if } j \in C \text{ and } j \neq s \\ b_s(C) & \text{if } j = s \\ 0 & \text{otherwise.} \end{cases}$$

Proposition C6. *A profile such that (a) $t_{ji}(C) = b_j(C)$ for any $j \neq s$ and (b) the outcome solves $\max_{C \in \mathcal{C}_s} \sum_i u_i(C)$ is a SPE.*

Proof. This profile is such that $\omega_j(C) = 0$ for any $j \neq s$ and any $C \in \mathcal{C}_s$. As such, any player $j \neq s$ is indifferent between any of their moves. The seed s has no incentive to deviate since her payoff $\omega_s(C)$ solves $\max_{C \in \mathcal{C}_s} \sum_i u_i(C)$. □

While agents may resort to contracts to implement welfare-maximizing communities, such contracts are highly dependent upon a well-functioning third-party enforcement mechanism. Indeed, note that the contracts described in Proposition C6 require community members to relinquish all their wealth to the member that recruited them into the community. It is immediate that in the absence of third-party enforcement, agents would renege from such contract ex-post and keep their wealth instead. Well-functioning third-party enforcement, in turn, goes against the spirit of some settings in which we expect communities to form. For instance, in the context of criminal organizations, we expect that the state is not able to enforce contracts between criminals.

Repeated game without contracts. We now consider the setting of a repeated game, in order to show that informal contracts may also support the implementation of welfare-maximizing communities.

At each time period $t \geq 0$, agents play the community-formation game Γ_t with heterogeneous preferences. Once the community-formation game is over with community C_t as an outcome, agents play a transfer game. That is, each agent i makes the vector of transfers t_i to all agents. Transfers are constrained such that

1. Non-community members may not make nor receive transfers; that is, $t_{ij} = 0$ if $i \notin C_t$ or $j \notin C_t$.
2. Transfers satisfy a budget constraint. In other words, they may not exceed the welfare produced by the community; that is, $\sum_{i,j} t_{ij} = \sum_i u_i(C_t)$

Once the transfer game is over, flow payoffs accrue, with

$$\omega_{it}(C_t) = \sum_j t_{ji} - t_{ij}$$

The stage game is repeated infinitely many times and agents discount the future with rate $\delta \in (0,1)$. Payoffs at time t write

$$U_{it} = (1-\delta) \sum_{t=\tau}^{\infty} \delta^{\tau - t} \omega_{i\tau}(C_\tau)$$

We show that if agents are sufficiently patient, this game admits an equilibrium profile in which an efficient community is implemented at each time period. While the result resembles a folk theorem, obtaining it is non-trivial, because the stage game is itself dynamic. Under a static stage game, minimax threats usually obtain cooperation. It is unclear whether those threats are credible in a dynamic stage game.

Note that for any seed s, the stage game admits a SPE profile σ_s that has (possibly inefficient) community $C_s \in \mathcal{C}_s$ as an outcome and such that no transfer occurs (i.e., $t_{ij} = 0$ for any $i \neq j$). Our candidate profile σ has an efficient community $C_s^* \in \mathcal{C}_s$ as an outcome. It rewards cooperation by using transfers to distribute the additional surplus generated by C_s^* relative to C_s in ways that make the members of C_s^* better off than under C_s, and deters defections by the threat of reverting back to σ_s. Formally:

Definition C10 (Candidate profile σ)**.** *For any seed $s \in N$, pick a strategy profile σ_s that is a SPE of the stage game with seed s and such that no transfer occurs (i.e., $t_{ij} = 0$ for any $i \neq j$). If the outcome C_s of σ_s is efficient, set $\sigma = \sigma_s$. Otherwise, construct a profile σ such that (i) its outcome is some community efficient community C_s^*, and (ii) all offers are accepted on path, and (iii) transfers satisfy $\omega_i(C_s^*) > \omega_i(C_s)$ for any $i \in C_s^*$. The off-path histories of σ are identical to those of σ_s. Following a deviation from σ, agents revert forever to σ_s every time seed s is picked.*

Proposition C7. *There is $\overline{\delta} \in (0, 1)$ such that for any $\delta \geq \overline{\delta}$, the profile σ is a SPE of the repeated game.*

Proof of Proposition C7. This proposition can be proven exactly as Proposition 2.7 in Ferrali (2020). □